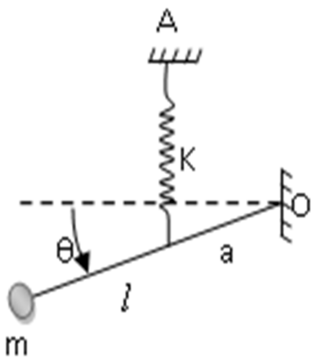
Solution de la série N°4

Exercice N°1 :



1. $U = U_k + U_m$

L'énergie potentielle de la tige égale à zéro car la masse de la tige est négligée.

$$dU_k = -\vec{F}_k d\vec{l} = -F_k dl \cos\left(\vec{F}_k(\uparrow), d\vec{l}(\downarrow)\right)$$

$$dU_k = -F_k dl \cos(180°)$$

$$\cos(180°) = -1$$

$$dU_k = +F_k dl = kxdx \ / \ F_k = kx, dl = dx$$

$$U_k = \int dU_k = \int_0^{x_0+x} kxdx = k\int_0^{x_0+x} xdx = k\left[\frac{x^2}{2}\right]_0^{x_0+x} = \frac{k}{2}\left[x^2\right]_0^{x_0+x}$$

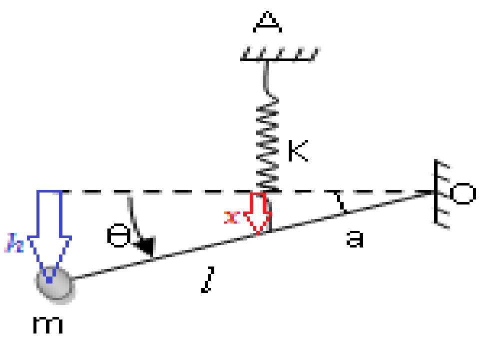
$$U_k = \frac{k}{2}\left[(x_0+x)^2 - 0\right]$$

$$U_k = \frac{k}{2}(x_0+x)^2 = \frac{k}{2}\left(x^2 + 2xx_0 + x_0^2\right) = \frac{k}{2}x^2 + kxx_0 + \frac{k}{2}x_0^2$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + \frac{k}{2}x_0^2 + cte \ / \ \frac{k}{2}x_0^2 = cte$$

Alors

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte$$



Avec

$$\sin\theta = \frac{x}{a} \Rightarrow x = a\sin\theta$$

$$U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}\left(a\sin\theta\right)^2 + k\left(a\sin\theta\right)x_0 + cte$$

$$dU_m = -\vec{F}_m d\vec{l} = -F_m dl\cos\left(\vec{F}_m\left(\downarrow\right), d\vec{l}\left(\downarrow\right)\right)$$

$$dU_m = -F_m dl\cos\left(0°\right)$$

$$\cos\left(0°\right) = +1$$

$$dU_m = -F_m dl = -mgdl \,/\, F_m = mg$$

$$U_m = \int dU_m = \int_0^h -mgdl = -mg\int_0^h dl = k\left[l\right]_0^h = -mg\left[h-0\right]$$

$$U_m = -mgh$$

$$\sin\theta = \frac{h}{l} \Rightarrow h = l\sin\theta \Rightarrow U_m = -mgl\sin\theta$$

Alors

$$U = U_k + U_m = \frac{k}{2}\left(a\sin\theta\right)^2 + k\left(a\sin\theta\right)x_0 - mgl\sin\theta + cte$$

$$U = \frac{k}{2}a^2\left(\sin\theta^2\right) + kax_0\left(\sin\theta\right) - mgl\left(\sin\theta\right) + cte$$

$$U = \frac{k}{2}a^2\left(\sin\theta^2\right) + \left(kax_0 - mgl\right)\left(\sin\theta\right) + cte$$

A faible amplitude $\sin\theta \approx \theta$ et $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U = \frac{k}{2}a^2\left(\theta^2\right) + \left(kax_0 - mgl\right)\left(\theta\right) + cte$$

Equilibre

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} = 0$$

$$\frac{\partial U}{\partial\theta} = \frac{k}{2}\times a^2\times 2\theta + \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2}\times a^2\times 2\left(\theta = 0\right) + \left(kax_0 - mgl\right) = \left(kax_0 - mgl\right)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow \left(kax_0 - mgl\right) = 0 \Rightarrow x_0 = +\frac{mgl}{ka}$$

C'est-à-dire que le ressort est allongé de $x_0 = +\frac{mgl}{ka}$

Alors

$U = \frac{k}{2}a^2\left(\theta^2\right) + cte$.

$$T = E_c = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$\dot{x} = l\dot{\theta}$$

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

Alors, le Lagrangien du système s'écrit :

$$L = E_c - E_p = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{k}{2}a^2\left(\theta^2\right) + cte$$

L'équation de mouvement pour des petites oscillations, est :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) + \left(\frac{\partial D}{\partial \dot{\theta}}\right) = F(t) = F_0 1\cos(\Omega t)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2 \times 2\dot{\theta} = ml^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{k}{2}a^2 \times 2\theta = -ka^2\theta$$

$$D = \frac{1}{2}\alpha\left(b\dot{\theta}\right)^2 = \frac{1}{2}\alpha\left(b^2\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\dot{\theta}^2$$

$$\frac{\partial D}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\alpha b^2\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right) = \frac{1}{2}\alpha b^2\left(2\dot{\theta}\right) = \alpha b^2\dot{\theta}$$

$$ml^2\ddot{\theta} + ka^2\theta + \alpha b^2\dot{\theta} = F_0 1\cos(\Omega t)$$

$$\begin{cases} \ddot{\theta} + \frac{ka^2}{ml^2}\theta + \frac{\alpha b^2}{ml^2}\dot{\theta} = \frac{F_0 1\cos(\Omega t)}{ml^2} \\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = F_\theta(\mathrm{t}) \end{cases}$$

$$\omega_0^2 = \frac{ka^2}{ml^2} \Rightarrow \omega_0 = \sqrt{\frac{ka^2}{ml^2}} = \frac{a}{l}\sqrt{\frac{k}{m}}$$

$$2\delta = \frac{\alpha b^2}{ml^2} \Rightarrow \delta = \frac{\alpha b^2}{2ml^2}$$

On a ; $b = 3l/4$ et $a = l/4$

$$\ddot{\theta}+\frac{k\left(\frac{l}{4}\right)^2}{ml^2}\theta+\frac{\alpha\left(\frac{3l}{4}\right)^2}{ml^2}\dot{\theta}=\frac{F_0\,1\cos(\Omega t)}{ml^2}$$

$$\ddot{\theta}+\frac{k\left(\frac{l^2}{16}\right)}{ml^2}\theta+\frac{\alpha\left(\frac{9l^2}{16}\right)}{ml^2}\dot{\theta}=\frac{F_0\cos(\Omega t)}{ml}$$

$$\ddot{\theta}+\frac{1}{16}\times\frac{k}{m}\theta+\frac{9}{16}\times\frac{\alpha}{m}\dot{\theta}=\frac{F_0\cos(\Omega t)}{ml}$$

$$\begin{cases}\ddot{\theta}+\frac{1}{16}\times\frac{k}{m}\theta+\frac{9}{16}\times\frac{\alpha}{m}\dot{\theta}=\frac{F_0\cos(\Omega t)}{ml}\\ \ddot{\theta}+\omega_0^2\theta+2\delta\dot{\theta}=F_\theta(\mathrm{t})\end{cases}$$

$$\omega_0^2=\frac{1}{16}\times\frac{k}{m}\Rightarrow\omega_0=\sqrt{\frac{1}{16}\times\frac{k}{m}}=\frac{1}{4}\sqrt{\frac{k}{m}}$$

$$2\delta=\frac{9}{16}\times\frac{\alpha}{m}\Rightarrow\delta=\frac{9}{32}\times\frac{\alpha}{m}$$

2. La résonnance du système

$$\omega_0=\frac{1}{4}\sqrt{\frac{k}{m}}$$

$$\delta=\frac{9}{32}\times\frac{\alpha}{m}$$

$$\frac{\delta}{\omega_0}=\xi<\frac{1}{\sqrt{2}}$$

$$\frac{\delta}{\omega_0}=\xi=\frac{\frac{9}{32}\times\frac{\alpha}{m}}{\frac{1}{4}\sqrt{\frac{k}{m}}}=\frac{9\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{\alpha}{m}}}{\frac{32}{4}\sqrt{\frac{k}{m}}}=\frac{9}{8}\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{\alpha}{m}}\times\sqrt{\frac{m}{k}}=\frac{9}{8}\times\frac{\alpha}{\sqrt{k\times m}}$$

Application Numérique :

On a $m=2\,kg$, $k=250\,N.m^{-1}$, $\alpha=5\,N.m^{-1}.\mathrm{s}^{-1}$

$$\frac{\delta}{\omega_0}=\xi=\frac{9}{8}\times\frac{\alpha}{\sqrt{k\times m}}=0.25<\frac{1}{\sqrt{2}}=0.7$$

$$\frac{\delta}{\omega_0}=\xi=0.7$$

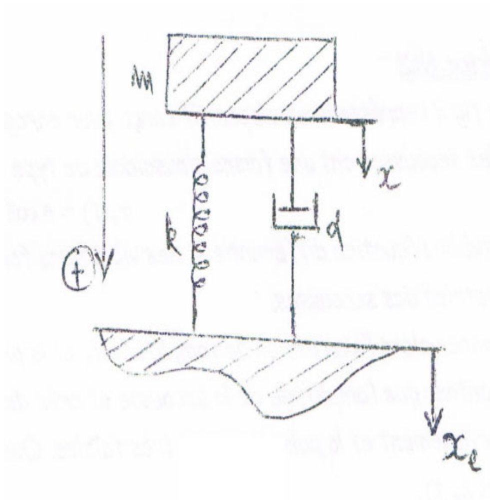
$$\Omega_R=\omega_0\sqrt{1-2\xi^2}\Rightarrow\Omega_R=10.45\,rad.s^{-\&}$$

3. L'amplitude de la résonnance :

$$\theta_{max} = \frac{F_0/ml}{\sqrt{\left(\omega_0^2 - \Omega_R^2\right) + 4\delta^2\Omega_R^2}} = 0.023\frac{F_0}{l}$$

θ_{max} : est proportionnelle a F_0 et inversement a l.

<u>Exercice N°2 :</u>



1. Le ressort subit une déformation de ses deux extrémités x du côté de la masse et x_e du côté du sol.

$$F_r = -k\left(x - x_e\right)$$

De même l'amortissement, qui en faisant déplacer ses deux extrémités avec des vitesses $(\dot{x}; \dot{x}_e)$ respectivement. : $f_r = -\alpha\left(\dot{x} - \dot{x}_e\right)$.

Donc :

$$m\ddot{x} = f_f + F_r = -k\left(x - x_e\right) - \alpha\left(\dot{x} - \dot{x}_e\right) = -kx + kx_e - \alpha\dot{x} + \alpha\dot{x}_e$$
$$m\ddot{x} + kx + \alpha\dot{x} = kx_e + \alpha\dot{x}_e$$

$$\ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = \frac{k}{m}x_e + \frac{\alpha}{m}\dot{x}_e \ avec \ x_e\left(t\right) = a\cos\left(\omega_e t\right)$$

$$\begin{cases} \ddot{x} + \dfrac{k}{m}x + \dfrac{\alpha}{m}\dot{x} = C\cos\left(\omega_e t + \theta\right) \\ \ddot{x} + \omega_0^2 x + 2\delta\dot{x} = C\cos\left(\omega_e t + \theta\right) \end{cases} \Rightarrow C = a\sqrt{\omega_0^4 + 4\delta^2\omega_e^2} \ et \ tg\,\theta = \frac{2\delta\omega_e}{\omega_0^2}$$

La solution de cette équation est $x\left(t\right) = x_h\left(t\right) + x_p\left(t\right)$

$$x_h(t) \underset{t\to 0}{\to} 0$$

$$x(t) \approx x_p(t) = A\cos(\omega_e t + \varphi)$$

$$A = \frac{a\sqrt{\omega_0^4 + 4\delta^2\omega_e^2}}{\sqrt{\left(\omega_0^2 - \omega_e^2\right)^2 + 4\delta^2\omega_e^2}}$$

$$tg\varphi = -\frac{2\delta\omega_e^3}{\omega_0^2\left(\omega_0^2 - \omega_e^2\right) + 4\delta^2\omega_e^2}$$

$$\ddot{\overline{x}} + \omega_0^2\overline{x} + 2\delta\dot{\overline{x}} = \overline{C}e^{j\omega_e t}$$

$$\overline{x} = \overline{A}e^{j\omega_e t}$$

$$\overline{A}\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)e^{j\omega_e t} = \overline{C}e^{j\omega_e t}$$

$$\overline{A} = \frac{\overline{C}}{\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)}$$

$$\left\|\overline{A}\right\| = \frac{C}{\sqrt{\left(\omega_0^2 - \omega_e^2 + j2\delta\omega_e\right)}}$$

$$tg\varphi = \frac{\operatorname{Im} A}{\operatorname{Re} A} = -\frac{2\delta\omega_e^3}{\omega_0^2\left(\omega_0^2 - \omega_e^2\right) + 4\delta^2\omega_e^2}$$

3. $\delta, \omega_e <<<$ Alors $4\delta^2\omega_e^2 << \omega_0^4 \; et \; \omega_e^2 << \omega_0^2 \Rightarrow A \approx a, tg\varphi = 0 \Rightarrow \varphi = 0$

Exercice N°3 :

1 . $Q = \frac{V_{c(\max)}}{E_0} = \frac{60}{3} = 20$.

2.

$$Q = \frac{1}{2\xi} = \frac{\omega_0}{2\delta} \Rightarrow \omega_0 = 2\delta Q$$

$$2\delta = \frac{R}{L} = \frac{75}{0.8\times 10^{-3}} = 93750\ s^{-1}$$

$$\omega_0 = 2\delta Q = 18.75\times 10^{+5}\ rad.s^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{L\omega_0^2} = 355.5\times 10^{-12}\,F.$$

3. $\Delta\omega = 2\delta = 93750\,s^{-1}$

$$\omega_1 \approx \omega_0 - \frac{\Delta\omega}{2} = 18.75\times 10^{+5} - \frac{93750}{2} = 18.25\times 10^{+5}\ rad.s^{-1}$$

$$\omega_2 \approx \omega_0 + \Delta\omega = 19.22\times 10^{+5}\ rad.s^{-1}$$

4.

$$\omega_1 \approx \omega_0 - \frac{\Delta\omega}{2} = 18.75\times10^{+5} - \frac{93750}{2} = 18.25\times10^{+5}\ rad.s^{-1}$$

$$\omega_2 \approx \omega_0 + \Delta\omega = 19.22\times10^{+5}\ rad.s^{-1}$$

$$P = \frac{1}{2}RI_0^2 = \frac{1}{2}75\left(30\times10^{-3}\right)^2 = 0.034\ watt$$