Chapter 5: Linear systems with several degrees of freedom

5.1 Degrees of freedom

The independent variables needed to describe a moving system are called degrees of freedom. If there are N independent variables qi, we write N Lagrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) = -\frac{\partial D}{\partial \dot{q}_1} + F_1(t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = -\frac{\partial D}{\partial \dot{q}_2} + F_2(t) \\ . \\ . \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_N}\right) - \left(\frac{\partial L}{\partial q_N}\right) = -\frac{\partial D}{\partial \dot{q}_N} + F_N(t) \end{cases}$$

5.1.1 Types of coupling

5.1.1.a Coupling by elasticity

The coupling between the two systems is through a spring (capacitance).

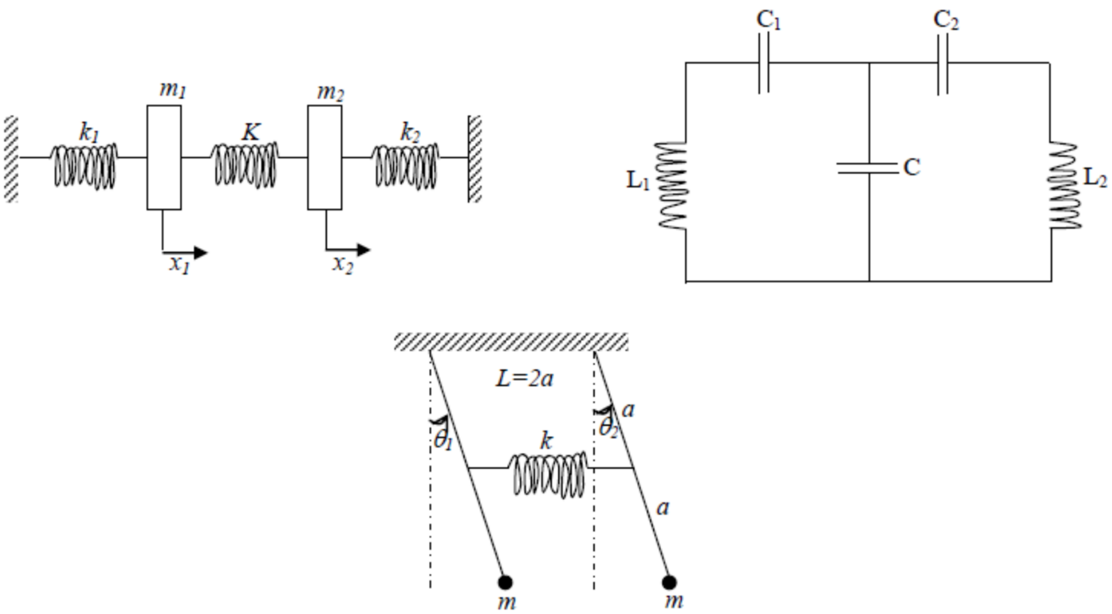


Figure 5.1: Different mechanical and electrical systems coupled by elasticity

5.1.1.b Couplage inertiel

The coupling between the two systems is through a mass (coil).

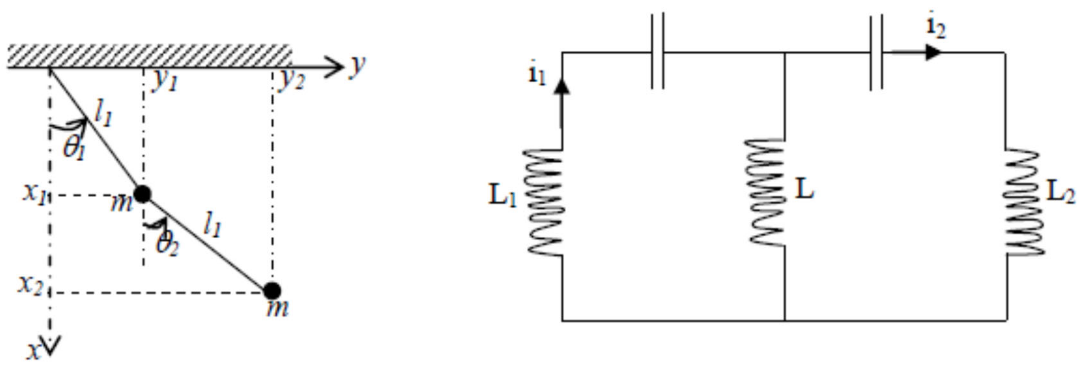


Figure 5.2 : Mechanical and electrical systems coupled by inertia

5.1.1.c Viscous coupling

The coupling between the two systems is through a damper (resistor).

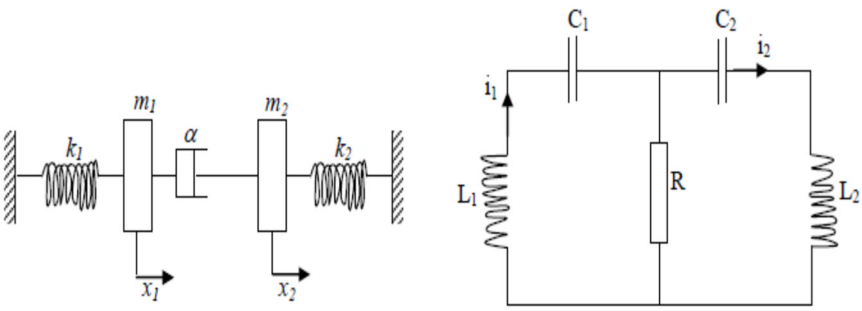


Figure 5.3: Viscous coupling of different mechanical and electrical systems

5.2 Free systems with two degrees of freedom

5.2.1 Equation of motion

Consider the free opposite system. The two independent variables are x_1 and x_2. k is called the coupling element.

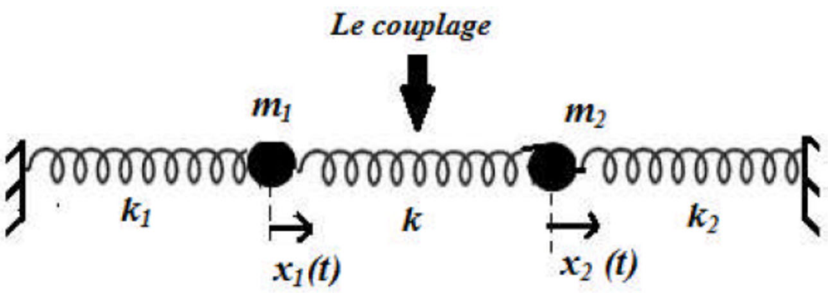


Figure 5.4: Oscillatory motion of a coupled system with two degrees of freedom

The kinetic energy is written as follows:

$$E_c = T = \sum_{i=1}^{2} \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

For the potential energy we have:

$$E_p = U = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} \sum_{i=1}^{2} k_i x_i^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k (x_1 - x_2)^2$$

Hence the Lagrangian is written:

$$L = T - U = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k (x_1 - x_2)^2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k (x_1^2 + x_2^2 - 2x_1 x_2)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 - \frac{1}{2} k (-2x_1 x_2)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 + k x_1 x_2$$

The two Lagrange equations are written: (For D=0, F=0: $system\ non$ damped and non $forced$)

The coupled differential system then becomes:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) - \left(\dfrac{\partial L}{\partial x_1}\right) = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) - \left(\dfrac{\partial L}{\partial x_2}\right) = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \left(\dfrac{\partial L}{\partial x_1}\right) = -k_1x_1 - kx_1 + kx_2 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2, \left(\dfrac{\partial L}{\partial x_2}\right) = -k_2x_2 - kx_2 + kx_1 \end{cases}$$

$$\begin{cases} m_1\ddot{x}_1 + (k_1+k)x_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + (k+k_2)x_2 - kx_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x}_1 + \dfrac{(k_1+k)}{m_1}x_1 - \dfrac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + \dfrac{(k+k_2)}{m_2}x_2 - \dfrac{k}{m_2}x_1 = 0 \end{cases}$$

5.2.2 Eigen modes

In normal mode (or eigen) the solution of the preceding equation is the form of a superposition of the two eigen modes, as follows:

$x_1 = A_1 \cos(\omega t + \varphi_1)$.

$x_2 = A_2 \cos(\omega t + \varphi_2)$.

A_1, A_2, φ, Depending on the initial conditions. To find ω, use the complex representation:

$$\begin{cases} x_1 = A_1\cos(\omega t + \varphi_1) \to \tilde{x}_1 = A_1 e^{j(\omega t + \varphi_1)} = \tilde{A}_1 e^{j\omega t} \\ x_2 = A_2\cos(\omega t + \varphi_2) \to \tilde{x}_2 = A_2 e^{j(\omega t + \varphi_2)} = \tilde{A}_2 e^{j\omega t} \end{cases}$$

$$\begin{cases} \tilde{x}_1 = \tilde{A}_1 e^{j\omega t} \\ \tilde{x}_2 = \tilde{A}_2 e^{j\omega t} \end{cases} \Rightarrow \begin{cases} \dot{\tilde{x}}_1 = j\omega\tilde{A}_1 e^{j\omega t} \\ \dot{\tilde{x}}_2 = j\omega\tilde{A}_2 e^{j\omega t} \end{cases} \Rightarrow \begin{cases} \ddot{\tilde{x}}_1 = (j\omega)^2\tilde{A}_1 e^{j\omega t} = -\omega^2\tilde{A}_1 e^{j\omega t} \\ \ddot{\tilde{x}}_2 = (j\omega)^2\tilde{A}_2 e^{j\omega t} = -\omega^2\tilde{A}_2 e^{j\omega t} \end{cases}$$

$$\begin{cases} \ddot{x}_1 + \dfrac{(k_1+k)}{m_1}x_1 - \dfrac{k}{m_1}x_2 = 0 \\ \ddot{x}_2 + \dfrac{(k+k_2)}{m_2}x_2 - \dfrac{k}{m_2}x_1 = 0 \end{cases} \Rightarrow \begin{cases} -\omega^2\tilde{A}_1 e^{j\omega t} + \dfrac{(k_1+k)}{m_1}\tilde{A}_1 e^{j\omega t} - \dfrac{k}{m_1}\tilde{A}_2 e^{j\omega t} = 0 \\ -\omega^2\tilde{A}_2 e^{j\omega t} + \dfrac{(k+k_2)}{m_2}\tilde{A}_2 e^{j\omega t} - \dfrac{k}{m_2}\tilde{A}_1 e^{j\omega t} = 0 \end{cases}$$

$$\begin{cases} -\omega^2\tilde{A}_1 e^{j\omega t} + \dfrac{(k_1+k)}{m_1}\tilde{A}_1 e^{j\omega t} - \dfrac{k}{m_1}\tilde{A}_2 e^{j\omega t} = 0 \\ -\omega^2\tilde{A}_2 e^{j\omega t} + \dfrac{(k+k_2)}{m_2}\tilde{A}_2 e^{j\omega t} - \dfrac{k}{m_2}\tilde{A}_1 e^{j\omega t} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(-\omega^2 + \dfrac{(k_1+k)}{m_1}\right)\tilde{A}_1 e^{j\omega t} - \left(\dfrac{k}{m_1}\right)\tilde{A}_2 e^{j\omega t} = 0 \\ \left(-\omega^2 + \dfrac{(k+k_2)}{m_2}\right)\tilde{A}_2 e^{j\omega t} - \left(\dfrac{k}{m_2}\right)\tilde{A}_1 e^{j\omega t} = 0 \end{cases}$$

$$\begin{cases} \left(-\omega^2 + \dfrac{(k_1+k)}{m_1}\right)\tilde{A}_1 - \left(\dfrac{k}{m_1}\right)\tilde{A}_2 = 0 \\ \left(-\omega^2 + \dfrac{(k+k_2)}{m_2}\right)\tilde{A}_2 - \left(\dfrac{k}{m_2}\right)\tilde{A}_1 = 0 \end{cases} \Rightarrow \begin{cases} \left(-\omega^2 + \dfrac{(k_1+k)}{m_1}\right)\tilde{A}_1 - \left(\dfrac{k}{m_1}\right)\tilde{A}_2 = 0 \\ -\left(\dfrac{k}{m_2}\right)\tilde{A}_1 + \left(-\omega^2 + \dfrac{(k+k_2)}{m_2}\right)\tilde{A}_2 = 0 \end{cases}$$

$$\begin{cases} (-\omega^2 + a)\tilde{A}_1 - b\tilde{A}_2 = 0 \\ -c\tilde{A}_1 + (-\omega^2 + d)\tilde{A}_2 = 0 \end{cases}$$

$$a = \frac{k+k_1}{m_1},\ b = \frac{k}{m_1},\ c = \frac{k}{m_2},\ d = \frac{k+k_2}{m_2}.$$

For the equation to be true without $\tilde{A}_1$ and $\tilde{A}_2$ both being zero, its characteristic determinant must be zero:

$$\Delta(\omega) = \begin{vmatrix} (-\omega^2 + a) & -b \\ -c & (-\omega^2 + d) \end{vmatrix} = (-\omega^2 + a)(-\omega^2 + d) - (-c)(-b) = 0$$

$$\Delta(\omega) = \omega^4 - \omega^2 d - \omega^2 a + ad - bc = \omega^4 - (d + a)\omega^2 + (ad - bc) = 0$$

This gives us the characteristic equation:

$$\omega^4 - (d + a)\omega^2 + (ad - bc) = 0$$

The two real and positive solutions ω_1 and ω_2 of this equation are called proper or normal pulsations. The smaller one is called the fundamental, the other is called the harmonic.

First eigenmode: For ω=ω_1, the system implies that:

$$\begin{cases} (-\omega^2 + a)\tilde{A}_1 - b\tilde{A}_2 = 0 \\ -c\tilde{A}_1 + (-\omega^2 + d)\tilde{A}_2 = 0 \end{cases} \Rightarrow \begin{cases} (-\omega^2 + a)\tilde{A}_1 = b\tilde{A}_2 \\ (-\omega^2 + d)\tilde{A}_2 = c\tilde{A}_1 \end{cases}$$

$\frac{\tilde{A}_1(1)}{\tilde{A}_2(1)} = \frac{-\omega_1^2+d}{c} > 0.$

The vibration is said to be in phase because the solution is written in this case

$$\begin{cases} x_{1(1)} = A_{1(1)} \cos(\omega_1 t + \varphi) \\ x_{2(1)} = A_{2(1)} \cos(\omega_1 t + \varphi) \end{cases}$$

Second eigenmode: For ω=ω2, the system implies that:

$\frac{\tilde{A}_1(2)}{\tilde{A}_2(2)} = \frac{-\omega_2^2+d}{c} < 0$. The vibration is said to be in phase opposition because the solution is written in this case:

$$\begin{cases} x_{1(2)} = A_{1(2)} \cos(\omega_2 t + \varphi) \\ x_{2(2)} = -A_{2(2)} \cos(\omega_2 t + \varphi) \end{cases}$$

In the general case, the system vibrates in a superposition of these two eigenmodes.

5.3 Forced system with two degrees of freedom

5.3.1 Motion equations

Consider the opposite system.

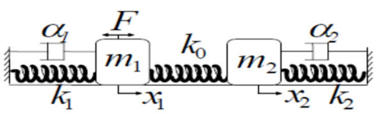


Figure 5.5: Oscillatory motion of a coupled system with two degrees of freedom

The kinetic energy is written as follows:

$$E_c = T = \sum_{i=1}^{2} \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

For the potential energy we have:

$$E_p = U = \frac{1}{2} k(x_1 - x_2)^2 + \frac{1}{2} \sum_{i=1}^{2} k_i x_i^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_0 (x_1 - x_2)^2$$

Hence the Lagrangian is written:

$$L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_0(x_1 - x_2)^2$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_0(x_1^2 + x_2^2 - 2x_1x_2)$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_0x_1^2 - \frac{1}{2}k_0x_2^2 - \frac{1}{2}k_0(-2x_1x_2)$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_0x_1^2 - \frac{1}{2}k_0x_2^2 + k_0x_1x_2$$

The two Lagrange equations are written: (For $D = \frac{1}{2}\alpha_1\dot{x}_1^2 + \frac{1}{2}\alpha_2\dot{x}_2^2$ and $F_0 \cos\omega t$)

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) + \frac{\partial D}{\partial \dot{x}_1} = +F \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) + \frac{\partial D}{\partial \dot{x}_2} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \left(\frac{\partial L}{\partial x_1}\right) = -k_1x_1 - k_0x_1 + k_0x_2, \frac{\partial D}{\partial \dot{x}_1} = \alpha\dot{x}_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2, \left(\frac{\partial L}{\partial x_2}\right) = -k_2x_2 - k_0x_2 + k_0x_1, \frac{\partial D}{\partial \dot{x}_2} = \alpha\dot{x}_2 \end{cases}$$

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_0)x_1 + \alpha_1\dot{x}_1 - k_0x_2 = F_0 \cos\omega t \\ m_2\ddot{x}_2 + (k_0 + k_2)x_2 + \alpha_2\dot{x}_2 - k_0x_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{x}_1 + \frac{(k_1 + k_0)}{m_1}x_1 - \frac{k_0}{m_1}x_2 + \frac{\alpha_1}{m_1}\dot{x}_1 = \frac{F_0}{m_1}\cos\omega t \\ \ddot{x}_2 + \frac{(k_0 + k_2)}{m_2}x_2 - \frac{k_0}{m_2}x_1 + \frac{\alpha_2}{m_2}\dot{x}_2 = 0 \end{cases}$$

5.3.2 Resonance and antiresonance

With D=0 and F≠0: $forced\ system$ but non damped) The permanent solution is:

$$\begin{cases} x_1 = A_1 \cos(\omega t + \varphi_1) \\ x_2 = A_2 \cos(\omega t + \varphi_2) \end{cases}$$

A_1, A_2, φ_2 depend on the excitation pulse ω and F_0. To find A_1, A_2, use the complex representation

When D= 0 :

$$\begin{cases} x_1 = A_1 \cos(\omega t + \varphi_1) \rightarrow \tilde{x}_1 = A_1e^{j(\omega t + \varphi_1)} = \tilde{A}_1e^{j\omega t} \\ x_2 = A_2 \cos(\omega t + \varphi_2) \rightarrow \tilde{x}_2 = A_2e^{j(\omega t + \varphi_2)} = \tilde{A}_2e^{j\omega t} \end{cases}$$

$$F(t) = F_0 \cos\omega t \rightarrow \tilde{F}(t) = F_0e^{j\omega t}$$

$$\begin{cases} \tilde{x}_1 = \tilde{A}_1e^{j\omega t} \\ \tilde{x}_2 = \tilde{A}_2e^{j\omega t} \end{cases} \Rightarrow \begin{cases} \dot{\tilde{x}}_1 = j\omega\tilde{A}_1e^{j\omega t} \\ \dot{\tilde{x}}_2 = j\omega\tilde{A}_2e^{j\omega t} \end{cases} \Rightarrow \begin{cases} \ddot{\tilde{x}}_1 = (j\omega)^2\tilde{A}_1e^{j\omega t} = -\omega^2\tilde{A}_1e^{j\omega t} \\ \ddot{\tilde{x}}_2 = (j\omega)^2\tilde{A}_2e^{j\omega t} = -\omega^2\tilde{A}_2e^{j\omega t} \end{cases}$$

$$\begin{cases} \ddot{x}_1 + \frac{(k_1 + k_0)}{m_1}x_1 - \frac{k_0}{m_1}x_2 = \frac{F_0}{m_1}e^{j\omega t} \\ \ddot{x}_2 + \frac{(k_0 + k_2)}{m_2}x_2 - \frac{k_0}{m_2}x_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\omega^2\tilde{A}_1e^{j\omega t} + \frac{(k_1 + k_0)}{m_1}\tilde{A}_1e^{j\omega t} - \frac{k_0}{m_1}\tilde{A}_2e^{j\omega t} = \frac{F_0}{m_1}e^{j\omega t} \\ -\omega^2\tilde{A}_2e^{j\omega t} + \frac{(k_0 + k_2)}{m_2}\tilde{A}_2e^{j\omega t} - \frac{k_0}{m_2}\tilde{A}_1e^{j\omega t} = 0 \end{cases}$$

$$\begin{cases} -\omega^2\tilde{A}_1e^{j\omega t}+\dfrac{(k_1+k_0)}{m_1}\tilde{A}_1e^{j\omega t}-\dfrac{k_0}{m_1}\tilde{A}_2e^{j\omega t}=\dfrac{F_0}{m_1}e^{j\omega t} \\ -\omega^2\tilde{A}_2e^{j\omega t}+\dfrac{(k_0+k_2)}{m_2}\tilde{A}_2e^{j\omega t}-\dfrac{k_0}{m_2}\tilde{A}_1e^{j\omega t}=0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(-\omega^2+\dfrac{(k_1+k_0)}{m_1}\right)\tilde{A}_1e^{j\omega t}-\left(\dfrac{k_0}{m_1}\right)\tilde{A}_2e^{j\omega t}=\dfrac{F_0}{m_1}e^{j\omega t} \\ \left(-\omega^2+\dfrac{(k_0+k_2)}{m_2}\right)\tilde{A}_2e^{j\omega t}-\left(\dfrac{k_0}{m_2}\right)\tilde{A}_1e^{j\omega t}=0 \end{cases}$$

$$\begin{cases} \left(-\omega^2+\dfrac{(k_1+k_0)}{m_1}\right)\tilde{A}_1-\left(\dfrac{k_0}{m_1}\right)\tilde{A}_2=\dfrac{F_0}{m_1} \\ \left(-\omega^2+\dfrac{(k_0+k_2)}{m_2}\right)\tilde{A}_2-\left(\dfrac{k_0}{m_2}\right)\tilde{A}_1=0 \end{cases} \Rightarrow \begin{cases} \left(-\omega^2+\dfrac{(k_1+k_0)}{m_1}\right)\tilde{A}_1-\left(\dfrac{k_0}{m_1}\right)\tilde{A}_2=\dfrac{F_0}{m_1} \\ -\left(\dfrac{k_0}{m_2}\right)\tilde{A}_1+\left(-\omega^2+\dfrac{(k_0+k_2)}{m_2}\right)\tilde{A}_2=0 \end{cases}$$

Case where: m_1=m_2=m and k_0=k_1=k_2=k.

By posing $\omega_0^2=\frac{k}{m}$, the equation becomes

$$\begin{cases} \left(-\omega^2+2\dfrac{k}{m}\right)\tilde{A}_1-\dfrac{k}{m}\tilde{A}_2=\dfrac{F_0}{m} \\ -\dfrac{k}{m}\tilde{A}_1+\left(-\omega^2+2\dfrac{k}{m}\right)\tilde{A}_2=0 \end{cases}$$

$$\begin{cases} (-\omega^2+2\omega_0^2)\tilde{A}_1-\omega_0^2\tilde{A}_2=\dfrac{F_0}{m}\ldots\ldots\ldots(1) \\ -\omega_0^2\tilde{A}_1+(-\omega^2+2\omega_0^2)\tilde{A}_2=0\ldots\ldots\ldots(2) \end{cases}$$

$$(2)\Rightarrow +(-\omega^2+2\omega_0^2)\tilde{A}_2=+\omega_0^2\tilde{A}_1$$

$$\tilde{A}_2=\frac{\omega_0^2}{(-\omega^2+2\omega_0^2)}\tilde{A}_1\ldots\ldots\ldots(3)$$

$we\ remplace\ (3)\ by\ (1)$

$$(-\omega^2+2\omega_0^2)\tilde{A}_1-\omega_0^2\frac{\omega_0^2}{(-\omega^2+2\omega_0^2)}\tilde{A}_1=\frac{F_0}{m}$$

$$(-\omega^2+2\omega_0^2)\tilde{A}_1-\frac{\omega_0^4}{(-\omega^2+2\omega_0^2)}\tilde{A}_1=\frac{F_0}{m}$$

$$\left((-\omega^2+2\omega_0^2)-\frac{\omega_0^4}{(-\omega^2+2\omega_0^2)}\right)\tilde{A}_1=\frac{F_0}{m}$$

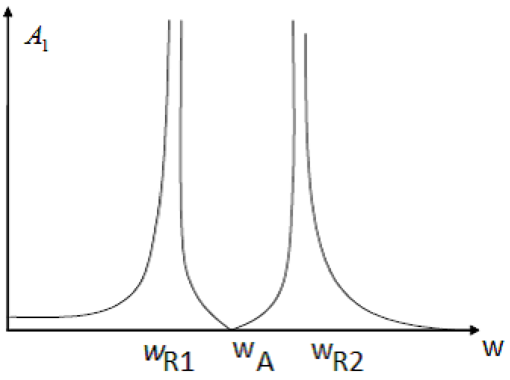
$$\left(\frac{(-\omega^2+2\omega_0^2)(-\omega^2+2\omega_0^2)-\omega_0^4}{(-\omega^2+2\omega_0^2)}\right)\tilde{A}_1=\frac{F_0}{m}$$

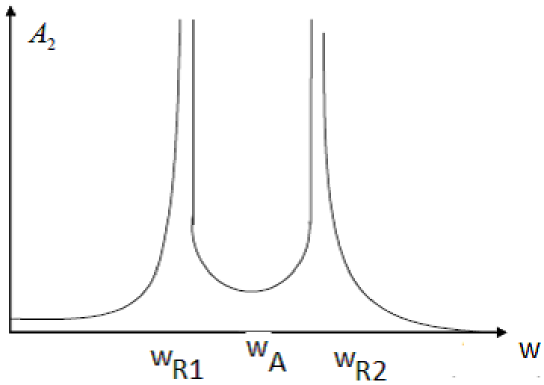
$$\left(\frac{(-\omega^2+2\omega_0^2)^2-\omega_0^4}{(-\omega^2+2\omega_0^2)}\right)\tilde{A}_1=\frac{F_0}{m}$$

$$\begin{cases} \tilde{A}_1=\dfrac{F_0}{m}\dfrac{(-\omega^2+2\omega_0^2)}{((-\omega^2+2\omega_0^2)^2-\omega_0^4)} \\ \tilde{A}_2=\dfrac{\omega_0^2}{(-\omega^2+2\omega_0^2)}\tilde{A}_1=\dfrac{F_0}{m}\dfrac{\omega_0^2}{(-\omega^2+2\omega_0^2)}\dfrac{(-\omega^2+2\omega_0^2)}{((-\omega^2+2\omega_0^2)^2-\omega_0^4)}=\dfrac{F_0}{m}\dfrac{\omega_0^2}{((-\omega^2+2\omega_0^2)^2-\omega_0^4)} \end{cases}$$

A_1=A_2=∞ when

- ✓ $\omega\cong\omega_0\cong\omega_{R1}$ (called first ***Resonance*** $pulsation$.)
- ✓ $\omega\cong\sqrt{3}\omega_0\cong\omega_{R2}$ ω=√3ω_0≡ω_{R2} (called second ***Resonance*** $pulsation$.)
- ✓ A_1=0 lorsque $\omega\cong\sqrt{2}\omega_0\cong\omega_A$ (called ***antiresonance*** $pulsation$.)





5.3.3 Input and transfer impedance

(With $D\neq0$ and $F\neq0$: Damped and forced system)

In electricity, the impedance is defined by $\tilde{z} = \frac{F}{\tilde{\imath}_1}$. By analogy, we define the mechanical impedance by $\tilde{z} = \frac{F}{v^{\tilde{\imath}}}$. $\tilde{z}_E = \frac{F}{\tilde{v}_1}$ is called input impedance. $\tilde{z}_T = \frac{F}{\tilde{v}_2}$ is called transfer impedance.

To find them we still use the complex representation:

$$F(t) = F_0 \cos \omega t \rightarrow \tilde{F}(t) = F_0 e^{j\omega t}$$

$$\begin{cases} x_1 = A_1 \cos(\omega t + \varphi_1) \rightarrow \tilde{x}_1 = A_1 e^{j(\omega t+\varphi_1)} = \tilde{A}_1 e^{j\omega t} \\ x_2 = A_2 \cos(\omega t + \varphi_2) \rightarrow \tilde{x}_2 = A_2 e^{j(\omega t+\varphi_2)} = \tilde{A}_2 e^{j\omega t} \end{cases}$$

$$\begin{cases} \tilde{v}_1 = j\omega \tilde{x}_1 \Rightarrow \tilde{x}_1 = \dfrac{\tilde{v}_1}{j\omega} \\ \tilde{v}_2 = j\omega \tilde{x}_2 \Rightarrow \tilde{x}_2 = \dfrac{\tilde{v}_2}{j\omega} \end{cases}$$

So

$$\begin{cases} \tilde{\ddot{x}}_1 = j\omega \tilde{v}_1 \\ \tilde{\ddot{x}}_2 = j\omega \tilde{v}_2 \end{cases}$$

$$\begin{cases} m_1 \tilde{\ddot{x}}_1 + (k_1 + k_0)\tilde{x}_1 + \alpha_1 \tilde{\dot{x}}_1 - k_0 \tilde{x}_2 = \tilde{F} \\ m_2 \tilde{\ddot{x}}_2 + (k_0 + k_2)\tilde{x}_2 + \alpha_2 \tilde{\dot{x}}_2 - k_0 \tilde{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} \left(j\omega m_1 + \dfrac{k_1}{j\omega} + \dfrac{k_0}{j\omega} + \alpha_1\right)\tilde{v}_1 - \dfrac{k_0}{j\omega}\tilde{v}_2 = \tilde{F} \\ \left(j\omega m_2 + \dfrac{k_2}{j\omega} + \dfrac{k_0}{j\omega} + \alpha_2\right)\tilde{v}_2 - \dfrac{k_0}{j\omega}\tilde{v}_1 = 0 \end{cases}$$

By posing $j\omega m_1 + \frac{k_1}{j\omega} + \alpha_1 = \tilde{z}_1, j\omega m_2 + \frac{k_1}{j\omega} + \alpha_2 = \tilde{z}_2, \frac{k_0}{j\omega} = \tilde{z}_0$ we obtain

$$\begin{cases} (\tilde{z}_1 + \tilde{z}_0)\tilde{v}_1 - \tilde{z}_0 \tilde{v}_2 = \tilde{F} \\ (\tilde{z}_2 + \tilde{z}_0)\tilde{v}_2 - \tilde{z}_0 \tilde{v}_1 = 0 \end{cases} \Rightarrow F = \left(\tilde{z}_1 + \tilde{z}_0 - \frac{\tilde{z}_0{}^2}{(\tilde{z}_2 + \tilde{z}_0)}\right)\tilde{v}_1 = \left(\tilde{z}_1 + \frac{\tilde{z}_2 \tilde{z}_0}{(\tilde{z}_2 + \tilde{z}_0)}\right)\tilde{v}_1$$

- The input impedance is $z_E = \frac{F}{\tilde{v}_1} = \left(\tilde{z}_1 + \frac{\tilde{z}_0 \tilde{z}_2}{(\tilde{z}_2 + \tilde{z}_0)}\right) \equiv \left(\frac{\tilde{z}_1 + \tilde{z}_0}{\tilde{z}_2}\right)$
- The transfer impedance is $z_T = \frac{F}{\tilde{v}_2} = \tilde{z}_1 + \tilde{z}_2 + \frac{\tilde{z}_1 \tilde{z}_2}{\tilde{z}_0}$