Chapter 4: Free linear damped systems with one degree of freedom

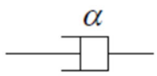
From the course of Dr Ghellab in French language

4.1 Damping force

A system subject to friction is said to be damped. The simplest friction is viscous friction. Viscous friction is of the form:

$$f = -\alpha v$$

α is a positive constant called the coefficient of friction and v is the speed of the moving body. In mechanics, damping is schematized by:



4.2 Lagrange equation of damped systems

If there is the friction $f = -\alpha\dot{q}$, the Lagrange equation becomes:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\alpha\dot{q}$.

By introducing the dissipation function $D = \frac{1}{2}\alpha\dot{q}^2$, we can write:

$f = -\alpha\dot{q} = -\frac{\partial D}{\partial \dot{q}}$. (In translation $D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$. In electricity $D = \frac{1}{2}Ri^2 = \frac{1}{2}R\dot{q}^2$). The Lagrange equation of damped systems is then written (where q=x, y, z, q, θ…)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = -\frac{\partial D}{\partial \dot{q}}$$

4.3 Equation of motion of damped systems

The equation of motion of linear damped systems by $f = -\alpha\dot{q}$ is of the form:

- The damped oscillation is defined as follows:

$$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$$

Where, δ is a positive coefficient and is called the damping factor. ω_0 is the proper pulsation.

$\frac{\omega_0}{2\delta} = Q$ is called the quality factor.

4.4 Solving the equation of motion

$$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$$

This equation is solved by the change of variable $q(t) = Ae^{rt} \Rightarrow \dot{q}(t) = Are^{rt} \Rightarrow \ddot{q}(t) = Ar^2e^{rt}$, the equation then becomes:

$\ddot{q}(t) + 2\delta\dot{q}(t) + \omega_0^2 q(t) = 0$

$$Ar^2e^{rt} + 2\delta Are^{rt} + \omega_0^2 Ae^{rt} = 0$$

$$Ae^{rt}(r^2 + 2\delta r + \omega_0^2) = 0$$

$$r^2 + 2\delta r + \omega_0^2 = 0$$

We calculate the discriminant Δ and we then obtain:

$$\Delta = (2\delta)^2 - 4\omega_0^2$$

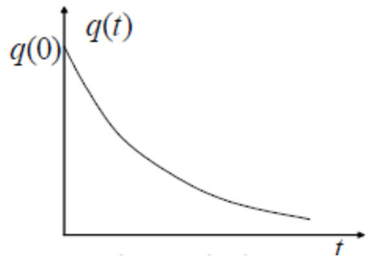
$$\Delta = 4\delta^2 - 4\omega_0^2$$

$$\Delta = 4(\delta^2 - \omega_0^2)$$

$$\sqrt{\Delta} = \sqrt{4(\delta^2 - \omega_0^2)} = 2\sqrt{(\delta^2 - \omega_0^2)}$$

There are three types of solution:

- ***Case where the system is strongly damped***: $\Delta > 0 \;\Rightarrow \delta > \omega_0$ and Q < 0.5



The solution of the differential equation is written as follows:

$$q(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \;avec\; r_{1,2} = \frac{-2\delta \pm 2\sqrt{\delta^2 - \omega_0^2}}{2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

$$q(t) = A_1 e^{\left(-\delta - \sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} = e^{-\delta t}\left(A_1 e^{\left(-\sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{\left(\sqrt{\delta^2 - \omega_0^2}\right)t}\right)$$

Where A_1 and A_2 are coefficients to be determined by the initial conditions

$$\begin{cases} q(t = 0) \\ \dot{q}(t = 0) \end{cases}$$

The system is said to have aperiodic motion.

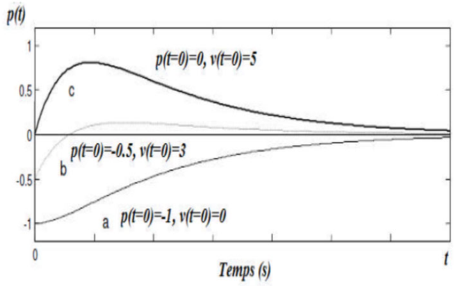
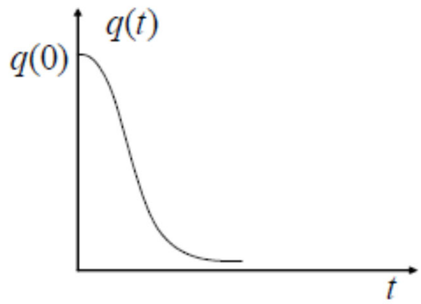


Figure 3.1 : Aperiodic damped movement

- **Case where damping is critical*:*** $\Delta = 0 \;\Rightarrow \delta = \omega_0$ et Q = 0.5



The solution of the equation is of the form:

$$q(t) = (A_1 t + A_2)e^{rt}$$

$$r_1 = r_2 = r = \frac{-2\delta}{2} = -\delta$$

$$q(t) = (A_1 t + A_2)e^{-\delta t}$$

Where A_1 and A_2 are coefficients to be determined by the initial conditions:

$$\begin{cases} q(t=0) \\ \dot{q}(t=0) \end{cases}$$

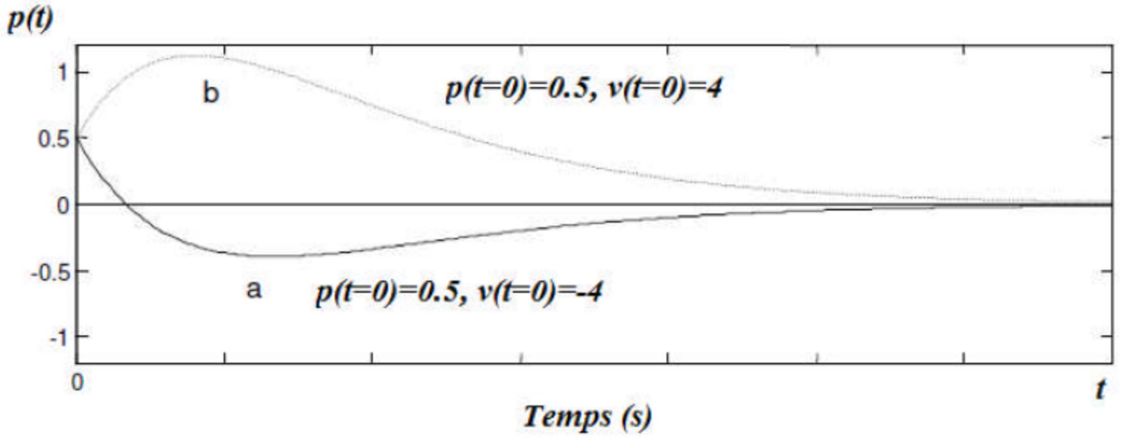
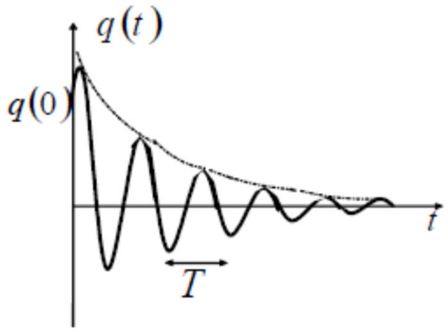


Figure 3.2 : Critical damped movement

- **Case where damping is low***:* $\Delta < 0 \Rightarrow \delta < \omega_0$ et Q > 0.5



Two complex solutions for the characteristic equation

$$\begin{cases} r_1 = -\delta - j\sqrt{\omega_0^2 - \delta^2} \\ r_2 = -\delta + j\sqrt{\omega_0^2 - \delta^2} \end{cases}$$

The resulting motion is $q(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t}$ that is:

$$q(t) = Ae^{-\delta t}\, cos(\omega t + \varphi)$$

Where, A et φ are constants to be determined by the initial conditions: $\begin{cases} q(t=0) \\ \dot{q}(t=0) \end{cases}$

The motion is said to be pseudo-periodic. $\omega = \sqrt{\omega_0^2 - \delta^2}$ is called pseudo-pulsation.

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}}$ is called pseudo-period.

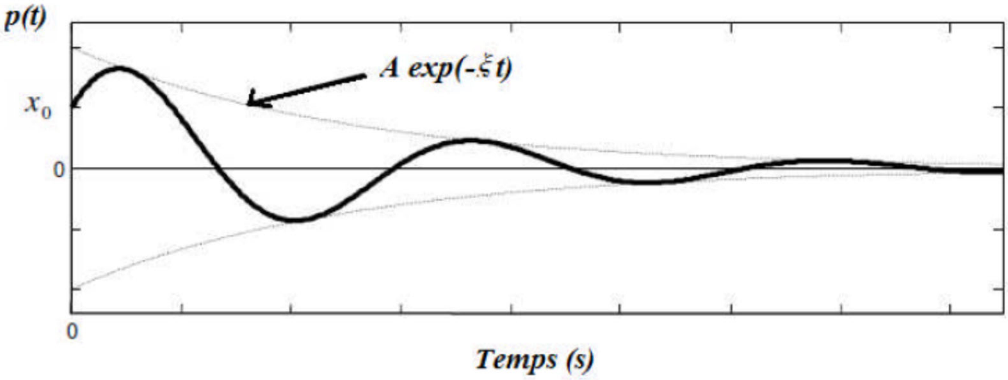


Figure 3.3: Damped oscillating motion

3.5 Logarithmic decrement

To evaluate the exponential decrease in the amplitude of the pseudo-periodic motion, we use the logarithm. The report

$D = Ln\frac{q(t)}{q(t+T)}$ or $D = \frac{1}{n}Ln\frac{q(t)}{q(t+nT)}$ is called the logarithmic decrement. Using the equation

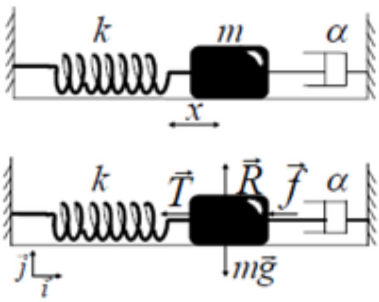
$q(t) = Ae^{-\delta t}\, cos(\omega t + \varphi)$, we find $\delta = Ln\frac{Ae^{-\delta t}}{Ae^{-\delta(t+T)}} \Rightarrow D = \delta\, T$

It should be noted that the system undergoes a total loss of energy due to the work of the friction forces.

$$dE_T(t) = -\alpha\dot{p}(t)^2 dt = -dW_{fr} \Rightarrow \Delta E_T + \Delta W_{fr} = 0$$

Examples

a) Consider the mass-spring system. Find the equation of motion first with the Lagrangian then with PFD.



Solution

• Using the Lagrangian:

The Lagrangian is:

$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$

$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$

The Lagrange equation is then written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = -\frac{\partial D}{\partial \dot{x}}$$

with $L = T - U \ = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + cte$

$$\frac{\partial L}{\partial \dot{x}} = \ \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{1}{2}m\frac{\partial}{\partial \dot{x}}(\dot{x}^2) = \frac{1}{2}m(2\dot{x}) = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\frac{d}{dt}(\dot{x}) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) = -\frac{1}{2}k\frac{\partial}{\partial x}(x^2) = -\frac{1}{2}k(2x) = -kx$$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2 \Rightarrow \frac{\partial D}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\alpha\dot{x}^2\right) = \frac{1}{2}\alpha\frac{\partial}{\partial \dot{x}}(\dot{x}^2) = \frac{1}{2}\alpha(2\dot{x}) = \alpha\dot{x}$$

So

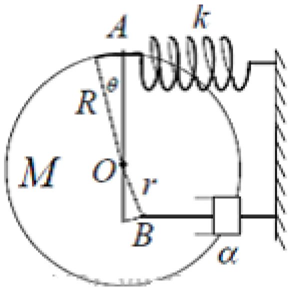
$$m\ddot{x} + kx = -\alpha\dot{x} \Rightarrow \begin{cases} \ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = 0 \\ \ddot{x} + \omega_0^2 x + 2\delta\dot{x} = 0 \end{cases}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} \ et \ 2\delta = \frac{\alpha}{m} \Rightarrow \delta = \frac{\alpha}{2m}$$

Using the PFD : $\sum\vec{F} = m\ \vec{a} \Rightarrow m\vec{g} + \vec{T} + \vec{R} + \vec{f} = m\ \vec{a} \Rightarrow -kx\vec{\imath} - mg\vec{\jmath} + R\vec{\jmath} - \alpha\dot{x}\vec{\imath} = m\ddot{x}\vec{\imath}$

By projection on $x'Ox$:$-kx - \alpha\dot{x} = m\ddot{x} \Rightarrow m\ddot{x} + kx + \alpha\dot{x} = 0 \Rightarrow \ddot{x} + \frac{k}{m}x + \frac{\alpha}{m}\dot{x} = 0$

b) Consider the disk-spring system. $\theta \ll 1$. Find the equation of motion if M=1kg, k=2 N/m, R=10cm, r=5cm, α=8Ns/m.



Using the Lagrangian:

$$T = E_c = T[Disque] = \frac{1}{2}J\dot{\theta}^2$$

$$J = \frac{1}{2}MR^2$$

$$T = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2$$

$$dU_k = -\vec{F}_k d\vec{l} = -F_k dl\, cos\left(\vec{F}_k(\uparrow), d\vec{l}(\downarrow)\right)$$

$$dU_k = -F_k dl\, cos(180°)$$

$$cos(180°) = -1$$

$$dU_k = +F_k dl = kxdx/F_k = kx, dl = dx$$

$$U_k = \int dU_k = \int_0^{x_0+x} kxdx = k\int_0^{x_0+x} xdx = k\left[\frac{x^2}{2}\right]_0^{x_0+x} = \frac{k}{2}[x^2]_0^{x_0+x}$$

$$U = U_k = \frac{k}{2}[(x_0+x)^2 - 0]$$

$$U = \frac{k}{2}(x_0+x)^2 = \frac{k}{2}(x^2 + 2xx_0 + x_0^2) = \frac{k}{2}x^2 + kxx_0 + \frac{k}{2}x_0^2$$

$$U = \frac{k}{2}x^2 + kxx_0 + \frac{k}{2}x_0^2 + cte/\frac{k}{2}x_0^2 = cte$$

$Alors$

$U = U_k = \frac{k}{2}x^2 + kxx_0 + cte$

Whith

$$sin\,\theta = \frac{x}{R} \Rightarrow x = R\,sin\,\theta$$

$$U = U_k = \frac{k}{2}x^2 + kxx_0 + cte = \frac{k}{2}(Rsin\,\theta)^2 + k(Rsin\,\theta)x_0 + cte$$

At low amplitude $sin\,\theta \approx \theta$

$$U = \frac{k}{2}R^2(\theta^2) + (kRx_0)(\theta) + cte$$

In the equilibrium

$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} = 0$

$$\frac{\partial U}{\partial \theta} = \frac{k}{2} \times R^2 \times 2\theta - (kRx_0)$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{\theta=0} : \theta = 0 \Rightarrow \frac{k}{2} \times R^2 \times 2(\theta = 0) - (kRx_0) = -(kRx_0)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = 0 \Rightarrow -(kRx_0) = 0 \Rightarrow x_0 = 0$$

So

$U = \frac{k}{2}R^2\theta^2 + cte.$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha\dot{x}^2$$

$$\dot{x} = r\dot{\theta} \Rightarrow D = \frac{1}{2}\alpha\left(r\dot{\theta}\right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = -\frac{\partial D}{\partial \dot{\theta}}$$

$$Avec\ L = T - U\ \ = U = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2 - \frac{k}{2}R^2\theta^2 + cte$$

$$\frac{\partial L}{\partial \dot{\theta}} = \ \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot{\theta}^2\right) = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right) = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(2\dot{\theta}\right) = \left(\frac{1}{2}MR^2\right)\dot{\theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\left(\frac{1}{2}MR^2\right)\dot{\theta}\right) = \left(\frac{1}{2}MR^2\right)\frac{d}{dt}\left(\dot{\theta}\right) = \left(\frac{1}{2}MR^2\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-\frac{1}{2}kR^2\theta^2\right) = -\frac{1}{2}kR^2\frac{\partial}{\partial \theta}(\theta^2) = -\frac{1}{2}kR^2(2\theta) = -kR^2\theta$$

$$D = \frac{1}{2}\alpha v^2 = \frac{1}{2}\alpha r^2\dot{\theta}^2 \Rightarrow \frac{\partial D}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\alpha r^2\dot{\theta}^2\right) = \frac{1}{2}\alpha r^2\frac{\partial}{\partial \dot{\theta}}\left(\dot{\theta}^2\right) = \frac{1}{2}\alpha r^2\left(2\dot{\theta}\right) = \alpha r^2\dot{\theta}$$

So

$$\left(\frac{1}{2}MR^2\right)\ddot{\theta} + kR^2\theta = -\alpha r^2\dot{\theta} \Rightarrow \begin{cases}\ddot{\theta} + \dfrac{kR^2}{\left(\frac{1}{2}MR^2\right)}\theta + \dfrac{\alpha r^2}{\left(\frac{1}{2}MR^2\right)}\dot{\theta} = 0\\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = 0\end{cases}$$

$$\Rightarrow \begin{cases}\ddot{\theta} + \dfrac{2k}{M}\theta + \dfrac{2\alpha r^2}{MR^2}\dot{\theta} = 0\\ \ddot{\theta} + \omega_0^2\theta + 2\delta\dot{\theta} = 0\end{cases}$$

$$\omega_0^2 = \frac{2k}{M} \Rightarrow \omega_0 = \sqrt{\frac{2k}{M}}\ \ et\ \ 2\delta = \frac{2\alpha r^2}{MR^2} \Rightarrow \delta = \frac{\alpha r^2}{MR^2}$$