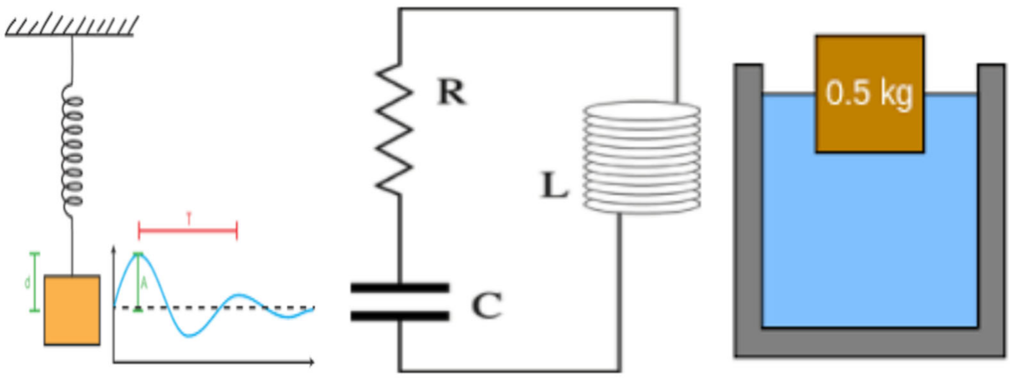
Chapter 2: Introduction to vibratory systems and LaGrange equations

1.1 Definition of an oscillation (Vibration)

Vibration is an oscillatory physical phenomenon of a moving object around its equilibrium position. Among the most varied mechanical movements, there are movements which are repeated: the beating of the heart, the movement of a seesaw, the alternating movement of the pistons of an internal combustion engine. All these movements have one thing in common: a repetition of the movement over a cycle.

Examples



a) Mass-spring b) Oscillating electric circuit c) cylinder floating in a liquid

A cycle is an uninterrupted sequence of movements or phenomena which are always renewed in the same order. Take as an example the four-stroke cycle of an internal combustion engine. A complete cycle comprises four stages (intake, compression, explosion, exhaust) which are repeated during an engine cycle.

1.2 Definition of a periodic motion

A periodic motion is a motion that repeats itself and each cycle of which is reproduced identically. The duration of a cycle is called period measured by the second and is defined as follows: $T_0 = \frac{2\pi}{\omega_0}$.

Where ω_0 is called the pulsation which is related to the frequency of the oscillations and is measured in rad.s^{-1}

$$\omega_0 = 2\pi f_0$$

Frequency is defined as the number of oscillations that occur per unit time t, and is measured in Hertz, $f_0 = \frac{1}{T_0}$

A particularly interesting periodic motion in the field of mechanics is that of an object which moves from its position of equilibrium and returns to it by performing a movement back and forth with respect to this position. This type of periodic motion is called oscillation or oscillatory motion. The oscillations of a mass connected to a spring, the movement of a pendulum or the vibrations of a stringed instrument are examples of oscillatory movements.

Mathematically, the periodicity is expressed by $f(t + T) = f(T)$.

1.3 Sinusoidal oscillatory motion

A periodic quantity is called Sinusoidal when it is of the form $q(t) = A\,sin(\omega t + \varphi)$.

The quantity q(t) is called the elongation (or the position) at the instant t.

A is the maximum elongation or the amplitude of the movement, it varies between –A and +A.

ω: The quantity is the pulsation of the movement and expressed in (rad/s).

$\omega t + \varphi$: is the instantaneous phase,

φ: the initial phase, corresponds to the phase at time = 0.

Note:

To facilitate the calculations, we transform the sinusoidal quantities into exponentials which are easier to manipulate. This is possible thanks to the Euler formula (1748).

$$cos\,\theta + i\,sin\,\theta = e^{i\theta}\ avec\ i^2 = -1$$

1.4 Physical modeling of a vibratory system:

Any mechanical system, including the most complex industrial machines, can be represented by models made up of a spring, a shock absorber and a mass. The human body, often referred to as "beautiful mechanics", is broken down in Figure 1.1 into several "spring-damper mass" subsystems representing the head, shoulders, rib cage and legs or feet.

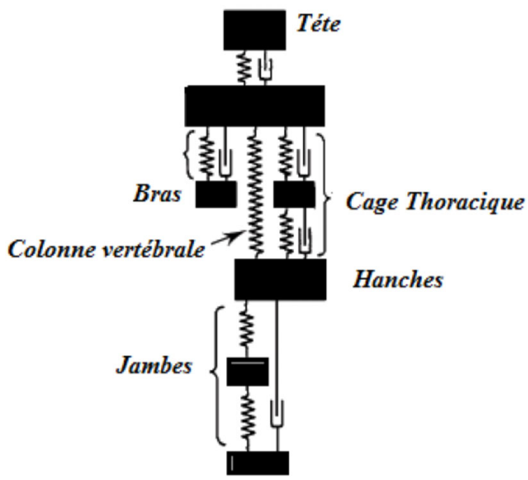


Figure 1.1: Mass-spring-damper modeling of humans.

To understand the vibratory phenomenon, all physical systems are associated with a "mass-spring" system which constitutes an excellent representative model for studying oscillations as follows, figure 1.2:

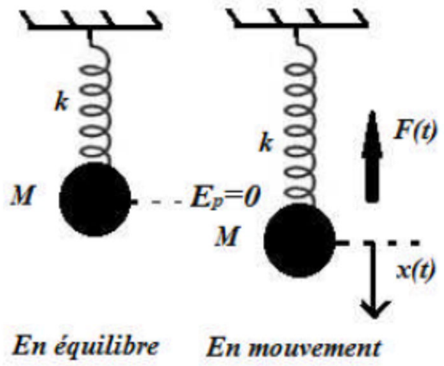


Figure 1.2: Mass-spring system

F(t) is called the restoring force which is proportional to the elongation x(t). The constant k is called the stiffness constant.

1.4.1 The representation of several springs

- There are two other configurations for the mass-spring system, figure 1.3:

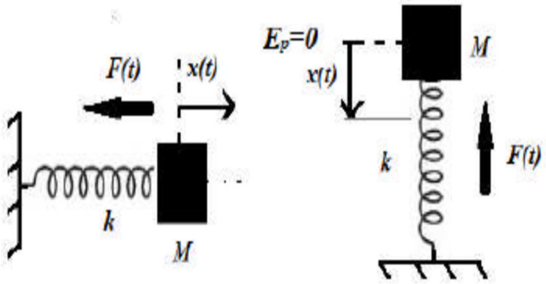


Figure 1.3: Configurations for the mass-spring system

The representation of several springs is presented in two cases:

✓ In parallel, we have figure 1.4:

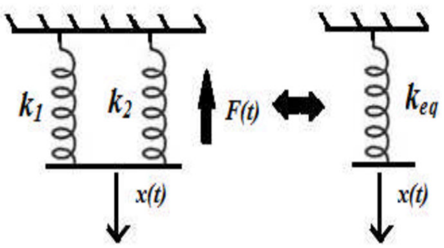


Figure 1.4: Springs in parallel

The equivalent stiffness is the sum of the stiffnesses k_1 and k_2 such that: $k_{eq} = k_1 + k_2$

✓ In series, we have figure 1.5:

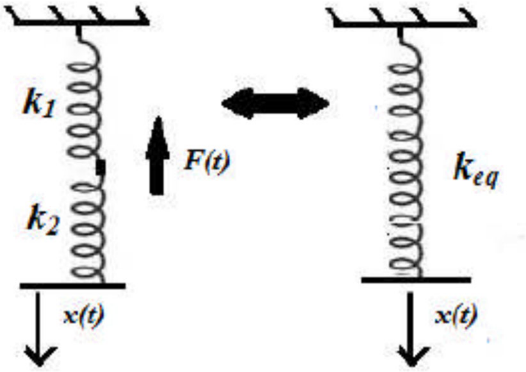


Figure 1.5: Springs in series

The equivalent stiffness for the constants $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ such that: $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$

An oscillating physical system is identified by the generalized coordinate $\boldsymbol{q}$ which is defined by the deviation from the position of stable equilibrium.

We define $\boldsymbol{n}$ the number of degrees of freedom by the number of independent motions of a physical system which determines the number of differential equations of motion.

1.5 Generalized coordinates of a physical system

We define generalized coordinates of a physical system as a set of real variables that do not all correspond to Cartesian coordinates (for example, angles, relative positions), allowing us to describe the system, particularly within the context of Lagrangian mechanics.

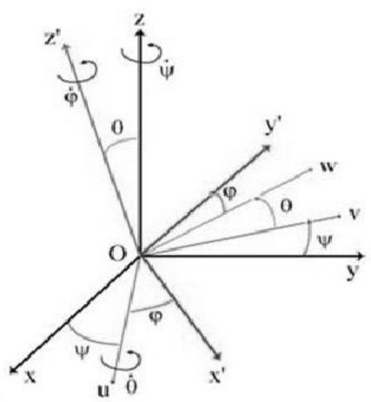
Generalized coordinates are not always assumed to be independent, and their advantage over Cartesian coordinates alone is the ability to select the best coordinates to represent the system while taking into account its constraints.

In the case of a pendulum, for example, it is advantageous to use the angle of the pendulum among the generalized coordinates.

The generalized coordinates are $n \leq 3N$, where N is the number of points allowing the system to be described and are often denoted: $q_1(t)$, $q_2(t)$, … … … . $q_n(t)$.

Example

- A free material point in space can be determined by these three (03) generalized coordinates: (x, y, z);
- A solid can be determined by 6 generalized coordinates:
• 03 coordinates relative to the center of gravity;
• 03 coordinates related to Euler angles (φ, ψ, θ).
- The generalized coordinates of a system P of material points and Q solid object are defined by: N= 3P + 6Q coordinates.



1.6 Degrees of freedom

The expression degree of freedom, in mechanics, is a concept covering the possibility of movement in space.

The degree of freedom is the number of independent generalized coordinates, necessary to configure all the elements of the system at any time.

There is a relation to calculate the number of the degrees of freedom:

For a mass point:

$$N_d = 3N_p - N_c$$

3 translations (x, y, z)

Where:

N_d: Degree of freedom

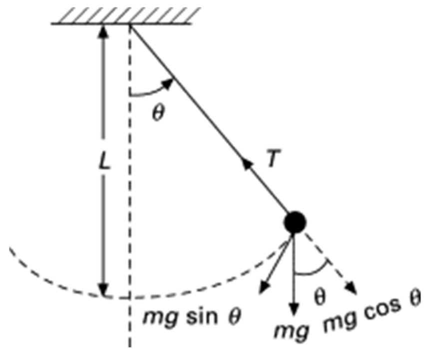
N_p: Number of generalized coordinates

N_c: Number of relations linking these coordinates together

Example

$Z=0$ and $x^2+y^2=l^2$

$N_d = 3N_p - N_c = 3 * 1 - 2 = 1$



For a solid object:

$$N_d = 6N_p - N_c$$

3 translations (x, y, z) and 3 rotations (φ, ψ, θ) = 6

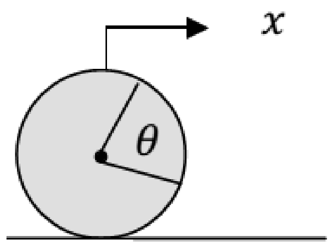
Example

A disk of mass m and radius r rolls without slipping on a horizontal plane.

φ= cte, ψ = cte, x=cte, y=cte and x=r θ

$N_d = 6N_p - N_c = 6 * 1 - 5 = 1$

one degree of freedom



1.7 Lagrange-Euler Method

The Lagrangian (L) of a dynamical system is a function of the dynamical variables which makes it possible to concisely write the equations of motion of the system. Its name comes from Joseph-Louis Lagrange, who established the principles of the process (from 1788).

Lagrangian mechanics was historically a reformulation of classical mechanics using the concept of Lagrangian. In this context, the Lagrangian is usually defined by the difference between the kinetic energy Ec = T and the potential energy Ep = V

$$L(q, \dot{q}) = T - U = E_c - E_p$$

where *L:* is the Lagrange of the system.

This method based on the principle of least action inspired by Fermat's principle of least time. Indeed, this method compares the actions corresponding to different possible trajectories and chooses the path for which the action is minimal.

We define the action of the system as a summation, between the time interval, $[t_0, t_1]$ along the trajectory of the system, of the difference between the kinetic energy and the potential energy.

$$\Gamma = \int_{t_0}^{t_1} L(q, \dot{q}) dt$$

The trajectory is determined by a variational method. This method results in the Euler-Lagrange equations which give trajectories on which the action is minimal.

- By applying the principle of least action, $\partial\Gamma = 0$, we obtain the Euler-Lagrange equation for a conservative system with several degrees of freedom (number n) as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \quad i = 1, \dots n$$

Where: L: is the Lagrangian function, q_i: is the generalized coordinate and $\dot{q}_i$: is the generalized velocity of the system.

In this case, the forces derive from a potential.

- For a dissipative (non-conservative) system with several degrees of freedom, the equation of motion determined as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = \overline{\sum |\vec{F}_{ext}|} \quad i = 1, \dots n$$

Where, $\vec{F}_{ext}$ are the external forces applied to the system. In this case, the external forces do not derive from a potential.

- Rayleigh dissipation function

In general, for a system with n degrees of freedom, the dissipation function for friction of the viscous type (low velocities) has the quadratic form of the generalized velocities:

$$D = \frac{1}{2} \sum_{i,j=1}^{n} \chi_{ij} \dot{q}_i \dot{q}_j$$

The q_i -component F_{q_i} of the friction force can then be written:

$$F_{q_i} = -\frac{\partial D}{\partial \dot{q}_i}$$

Finally, the Euler-Lagrange equation is then written:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = -\frac{\partial D}{\partial \dot{q}_i} \quad i = 1, \dots n$$

- Case of a time-dependent external force

the more general case of a time-dependent external force acting on a system subject to frictional forces 'deriving' from a dissipation function D.

F_{ext} the q-component of the external force.

In this case the Euler-Lagrange equation of a system with several degrees of freedom can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = -\frac{\partial D}{\partial \dot{q}_i} + F_{q_i}^{ext} \quad i = 1, \dots n$$

Where, $F_{q_i}^{ext}$ is the external force that causes the generalized coordinate qi to vary.