Chapter 1: Mathematical and Physical background (remind)

1.1 Mathematical background (rappel)

1.1.1 Taylor series

Let the function $f(x)$ derivable till the n+1 order at the neighborhood x_0 .

The function $f(x)$ accepts to develop in the neighborhood of x_0 according to the following Taylor's law:

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^n(x_0) + R_n(x)$$

Where, $R_n(x)$ is the rest of the development.

where $f^n(x_0)$ denotes the n^{th} derivative. (The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and 0! are both defined to be 1)

If $f(x)$ is infinitely differentiable $\Longrightarrow$ $f(x) = f(x_0) + \sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n!}f^n(x_0)$

When $x_0 = 0$, the series is also called a Maclaurin series $\Longrightarrow$ $f(x) = f(0) + \sum_{n=1}^{\infty}\frac{x^n}{n!}f^n(0)$

Example

developing of sin x, cos x and ex at the nationhood x_0

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!}$$

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots + \frac{x^n}{(n)!}$$

1.1.2 Periodic functions:

A periodic function is a function that repeats its values at regular intervals. Any function that is not periodic is called aperiodic.

A function f is said to be periodic if, for some nonzero constant T,

$$f(x) = f(x+T) = f(x+2T) = \cdots = f(x+nT)$$

n is an integer

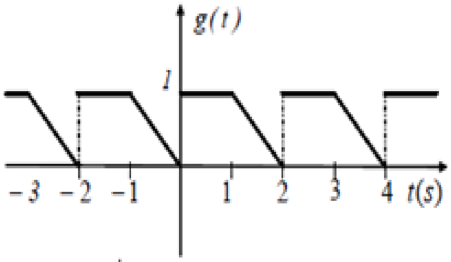
T is the period of the function and it is the smallest amount that achieves the above equality, which could be time (t) or distance (x).

For time: the period is T(s), the frequency is $f, \nu = \frac{1}{T}$ (Hz, s^{-1}), the pulsation is $\omega = 2\pi\nu$(rad s^{-1})

For distance: the period is λ(m), the frequency is $\sigma = \frac{1}{\lambda}$ (m^{-1}), the pulsation is $k = 2\pi\sigma$(rad m^{-1})

Example

Consider the periodic function f (t) shown in this figure.



$$T = 2s.\ f = \frac{1}{T} = 0.5Hz.$$

$$\omega = 2\pi f = \frac{\pi rad}{s}$$

1.1.3 Fourrier series

A periodic quantity can be expressed as a sum of sines and cosines, which are easier to manipulate physically and mathematically. This sum is known as the Fourier series (1807).

The Fourier series of a periodic function f (t) of period T, is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega\ t) + \sum_{n=1}^{\infty} b_n \sin(n\omega\ t)$$

- ✓ a_0, a_n and b_n are called the Fourier coefficients.
- ✓ The pulsation ω= $2\pi/T$ is called the fundamental pulsation.
- ✓ The upper pulses $n\omega$ (multiple of ω) are called the harmonics.
- ✓ The Fourier coefficients are defined by:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt, \quad a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega\ t)\,dt, \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega\ t)\,dt$$

➢ Case of even and odd functions

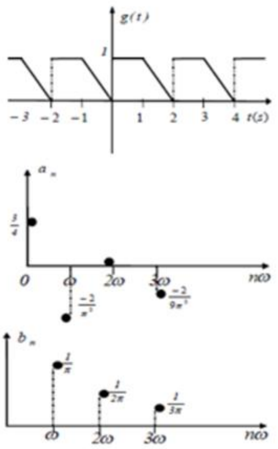
- Even functions: A function is said to be even if $f(-t) = f(t)$.

In the Fourier series of even functions, there are only the cosine terms and sometimes the constant a_0 which is the average value of the function: b_n =0.

- Odd functions: A function is said to be odd if $f(-t) = -f(t)$

In the Fourier series of odd functions, there are only sine terms: $a_0 = a_n = 0$.

Example



1. The period of the function is T=2s.

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\left[\int_0^1 (1)dt + \int_1^2 (-t+2)\,dt\right] = \frac{3}{4}.$$

$$a_n = \frac{2}{T}\int_0^T f(t)\,cos\frac{(2\pi n\ t)}{T}dt = \frac{2}{2}\left[\int_0^1 (1)\,cos(n\pi\ t)\,dt + \int_1^2 (-t+2)\,cos(n\pi\ t)\,dt\right]$$

$$= \frac{cos(n\pi)-1}{n^2\pi^2}$$

$$a_n = \frac{(-1)^n-1}{n^2\pi^2} = \begin{cases} 0 & if\ n \text{ is even} \\ \dfrac{-2}{n^2\pi^2} & si\ n \text{ is odd} \end{cases}$$

$b_n = \frac{2}{T}\int_0^T f(t)\,sin\frac{(2\pi n\ t)}{T}dt = \frac{2}{2}\left[\int_0^1 (1)\,sin(n\pi\ t)\,dt + \int_1^2 (-t+2)\,sin(n\pi\ t)\,dt\right] = \frac{1}{\pi n}$Donc the Fourier series is $f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi^2}cos(n\omega t) + \sum_{n=1}^{\infty}\frac{1}{n\pi}sin(n\omega t)$

The spectrum of $f(t)$ will be taken to be the graph of a_n and b_n as a function of $n\omega$.

1.1.4 Sinusoidal functions:

A periodic function is said to be Sinusoidal when it has the form $f(t) = A\,sin(\omega t + \varphi)$.

A is called amplitude, ω: the pulsation, φ: the initial phase. Among the studied physical quantities of oscillating systems, we find:

- ✓ The displacement x.
- ✓ The angle θ.
- ✓ The charge q.
- ✓ The current i.
- ✓ The voltage U.
- ✓ The field E.

1.1.5 The complex representation

To facilitate the calculations, we transform the sinusoidal quantities into exponentials which are easier to manipulate. This is possible thanks to the Euler formula (1748).

$$\cos\theta + j\sin\theta = e^{j\theta} \; avec \; j^2 = -1$$

Examples

Consider a capacitor and a current $i(t) = I_0 \cos\omega\, t$.

Find the complex impedance $\tilde{z}_c = \frac{\tilde{u}_c}{\tilde{i}}$. *Rappel :* $V_c = \frac{q}{c} = \frac{\int idt}{c}$ *car* $i = \frac{dq}{dt}$

$\begin{cases} i(t) = I_0 \cos\omega\, t \rightarrow \tilde{\imath}(t) = I_0 e^{j\omega t} \\ V_c(t) = \frac{\int i(t)dt}{c} \rightarrow \tilde{V}_c(t) = \frac{\int I_0 e^{j\omega t} dt}{c} = \frac{I_0 e^{j\omega t}}{jc\omega} = \frac{\tilde{\imath}(t)}{jc\omega} \end{cases} \Rightarrow \tilde{z}_C = \frac{\tilde{V}_C}{\tilde{\imath}} = \frac{\tilde{\imath}(t)}{jc\omega\tilde{\imath}} = \frac{1}{jc\omega}c)$ Let L be a coil and a current $i(t) = I_0 \cos\omega t$.

Find the complex impedance $\tilde{z}_L = \frac{\tilde{V}_L}{\tilde{\imath}}$. Rappel : $\tilde{V}_L = L\frac{di}{dt}$.

$$\begin{cases} i(t) = I_0 \cos\omega\, t \rightarrow \tilde{\imath}(t) = I_0 e^{j\omega t} \\ V(t) = \frac{di}{dt} \rightarrow \tilde{V}_L(t) = L\frac{d\tilde{\imath}}{dt} = L\frac{d\left(I_0 e^{j\omega t}\right)}{dt} = jL\omega I_0 e^{j\omega t} = jL\omega\tilde{\imath} \end{cases} \Rightarrow \tilde{z}_L = \frac{\tilde{V}_L}{\tilde{\imath}} = \frac{jL\omega\tilde{\imath}}{\tilde{\imath}} = jL\omega$$

1.1.6 Superposition of periodic functions

The addition of two or more functions of the same nature is called superposition.

1.1.6.1 Sinusoidal functions of the same pulsation

The superposition of two sinusoidal functions of the same pulsation ω is a sinusoidal function of pulsation ω.

Example

Consider the two sinusoidal functions:

$$f_1(t) = \sqrt{2}\cos\left(3t - \frac{\pi}{4}\right) \; et \; f_2(t) = \sqrt{2}\sin(3t + \pi)$$

$$\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right) + \sqrt{2}\sin(3t + \pi) \;\rightarrow \sqrt{2}e^{j\left(\omega t - \frac{\pi}{4}\right)} + e^{j\left(\omega t + \pi - \frac{\pi}{2}\right)} = e^{j\omega t}\left(\sqrt{2}e^{j\left(-\frac{\pi}{4}\right)} + e^{j\left(\frac{\pi}{2}\right)}\right)$$

$$= 1 \times e^{j\omega t} = \cos(\omega t)$$

So : A=1. φ=0.

1.1.6.2 Sinusoidal functions of the same amplitudes

The superposition of two sinusoidal functions of the same amplitude is a sinusoidal function with modulated amplitude if the two pulsations are different.

Example

Consider the two sinusoidal functions:

$$g_1(t) = a\cos(\omega_1 t) \;\; and \;\; g_2(t) = a\sin(\omega_2 t)$$

The superposition is: $g_1(t) = (a\cos(\omega_1 t)) \; + \; g_2(t) = (a\sin(\omega_2 t))$

$$g(t) = g_1(t) + g_2(t) = 2a\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

1.1.6.1 Any sinusoidal functions

The superposition of two sinusoidal quantities of different pulsations ω1 and ω2 will only be a periodic quantity if the ratio between their period is a rational number: $\frac{T_1}{T_2} = \frac{n}{m}$ The resulting period is the least common multiple: $T=mT_1=nT_2$.

Example

Consider the two sinusoidal functions: $g_1(t) = 5\cos(5t+2)$ *et* $g_2(t) = 2\cos(7t+3)$

Their superposition is : $5\cos(5t+2) + 2\cos(7t+3)$.

As $\frac{T_1}{T_2} = \frac{2\pi/5}{2\pi/7} = \frac{n}{m}$ is a rational number.

(n=7, m=5), the resulting *superposition* is a *periodic* function of $T = m\left(2\pi/5\right) = n\left(2\pi/7\right) = 2\pi s$.

1.1.7 Linear second order differential equations with constant factors

A differential equation is an equation that relates one or more unknown functions and their derivatives. The most general form is:

$$y = f(\dot{y}, \ddot{y}, \dddot{y}, \dots\dots, y^{(n)})$$

Differential equations are described by their order, determined by the term with the highest derivatives.

An equation containing only first derivatives is a first-order differential equation, an equation containing the second derivative is a second-order differential equation, and so on.

A linear differential equation is one that does not contain any powers (greater than one) of the function or its derivatives.

The general second-order differential equation is as follow:

$$a\ddot{y} + b\dot{y} + cy = f(x)$$

Here a, b and c are constants.

This equation is homogeneous, if it is without second member $f(x) = 0$ and it is inhomogeneous if $f(t) \neq 0$.

If $a = 0$ the equation will be a first-order differential equation.

1.1.7.1 Homogeneous second order differential equation

To solve the second order homogeneous linear differential equation

$$a\ddot{y} + b\dot{y} + cy = 0$$

in which a, b and c are constants, start with the assumption that

$$y(x) = e^{rx}$$

where r is a constant to be determined, plugging this formula into our differential equation, and seeing if a constant r can be determined.

$$a\ddot{y} + b\dot{y} + cy = 0$$

yields

$$a\frac{d^2}{dx^2}[e^{rx}] + b\frac{d}{dx}[e^{rx}] + c[e^{rx}] = 0$$

$$\Rightarrow ar^2[e^{rx}] + br[e^{rx}] + c[e^{rx}] = 0$$

$$\Rightarrow e^{rx}[ar^2 + br + c] = 0$$

Since e^{rx} can never be zero, we can divide it out, leaving us with the algebraic equation

$$ar^2 + br + c = 0$$

This latter polynomial equation is called the characteristic equation for differential equation. The solution of the characteristic equation can always be obtained via the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice how the nature of the value *r* depends strongly on the value under the square root, $b^2 - 4ac$. There are three possibilities:

1- If $\Delta = b^2 - 4ac > 0$ we have two distinct real values for *r* , $\Rightarrow$

$r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

and the solution takes the form:

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

2- If $\Delta = b^2 - 4ac = 0$, then

$r_1 = r_2 = \frac{-b-\sqrt{0}}{2a}$ $\Rightarrow r = \frac{-b}{2a}$

we only have one real root for our characteristic equation

and the solution takes the form:

$$y_h(x) = (C_1 x + C_2)e^{rx}$$

3- If $\Delta = b^2 - 4ac < 0$ then the quantity under the square root is negative, and, thus, this square root gives rise to an *imaginary number*. To be explicit,

$$\sqrt{b^2 - 4ac} = \sqrt{-1.\,|b^2 - 4ac|} = i\sqrt{|b^2 - 4ac|}$$

where " $i^2 = -1$. Thus, in this case, we will get two distinct complex roots, $r_1 and\ r_2$ with

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{|b^2 - 4ac|}}{2a} = \frac{-b}{2a} \pm i\frac{\sqrt{|b^2 - 4ac|}}{2a}$$

and the solution takes the form:

$$y_h(x) = Ce^{-\delta t}cos(\omega_a t + \varphi)$$

1.1.7.2 Inhomogeneous second order differential equation

$$a\ddot{y} + b\dot{y} + cy = f(x)$$

The general solution is taking the form

$$y_g(x) = y_h(x) + y_p(x)$$

Where:

$y_h(x)$ is the solution of the homogenous equation

$y_p(x)$ is the special solution with the availability of the second member of the equation

Generally, $y_p(x)$ takes the form of the function $f(x)$.

1.2 Physical background (rappel)

1.2.1 Mechanical system

A mechanical system is a system of elements (masses, springs, ….) that interact on mechanical principles.

1.2.1.1 Conservative system

A system in which there is no dissipation of energy so that the total energy remains constant with time.

1.2.1.2 Dissipative (non-conservative) system

1.2.4 Total energy of a mechanical system

The total energy of a mechanical system is defined by the sum of the kinetic and potential energies.

1.2.4.1 Kinetic energy of a system

The kinetic energy of a mechanical system is written under the form:

$$T = E_C = \sum_{n>1} \frac{1}{2} m_i \dot{q}_1^2$$

- For a point mass :

$$T = E_C = \sum_{n>1} \frac{1}{2} m_i v_i^2$$

- For a solid object
  - ✓ ***In translation***

$$T_{trans} = E_{C_{trans}} = \frac{1}{2} M v_G^2$$

G is the center of gravity

  - ✓ ***In rotation***

$$T_{rot} = E_{C_{rot}} = \frac{1}{2} I_G \dot{\theta}^2$$

I_G is the moment of inertia

  - ✓ ***In translation and rotation***

$$T = T_{trans} + T_{rot} = \frac{1}{2} M v_G^2 + \frac{1}{2} J_G \dot{\theta}^2$$

Moment of inertia: $J_G = \int_0^M r^2 \, dm$

Example

For a material point of mass m, which rotates around an axis at a distance r, $J_G = m\, r^2$

Find the moment of inertia of a circle

Huygens-Steiner Theorem

The moment of inertia of a body about an arbitrary axis is equal to the sum of the moment of inertia of a body about an axis passing through the center of mass parallel to an arbitrary axis and the product of the mass of the body by the square of the distance between the axes.

$$J = J_G + md^2$$

1.2.4.2 Potential energy of a system

The potential energy of a mechanical system is written from limited Taylor's development under the form:

$$E_p = U = U(0) + \left.\frac{\partial U}{\partial q}\right|_{q=0} q + \frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} q^2 + \frac{1}{6}\left.\frac{\partial^3 U}{\partial q^3}\right|_{q=0} q^3 + \dots\dots\dots + \frac{1}{n!}\left.\frac{\partial^n U}{\partial q^n}\right|_{q=0} q^n$$

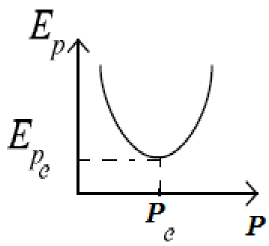
- The value *q=0* corresponds to the equilibrium position of the system characterized by

$$\left.\frac{\partial U}{\partial q_i}\right|_{q=0} = 0$$

There are two types of equilibrium:

- Stable equilibrium:
  - In this case, the necessary condition is that:

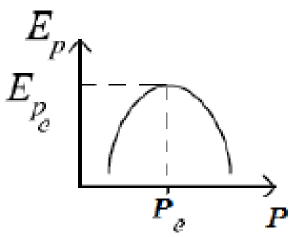
$$\frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} > 0.$$



Stable equilibrium

- Unstable equilibrium:
  - In this case, the necessary condition is that:

$$\frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} < 0.$$



Unstable equilibrium

- Oscillatory motion is said to be linear if this difference is infinitesimal. For this purpose, the potential energy takes the quadratic form as a function of the deviation from the equilibrium position represented as follows:

$$U = E_p = \frac{1}{2}\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} q^2$$

The constant $\frac{\partial^2 U}{\partial q^2}$ is called the recall constant.

Thereby; the restoring force takes the linear form as a function of the elongation and opposite to the motion such that:

$$\vec{F}(t) = -\left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0} \vec{q}$$

1.2.5 Calculation of the motion equation

The equation of motion for a conservative system can be determined by three methods:

1.2.5.1 Principle of total energy conservation

$$E_T = E_c + E_p = T + U = Constant \quad \Rightarrow \quad \frac{dE_T}{dt} = 0$$

Where E_T est appelée l'énergie totale du système.

1.2.5.2 Newton's Law of Dynamics

$$\sum_{n>1} \vec{F}_i = m_i \, \vec{a}_i$$

where $\vec{a}_i$ is called the acceleration of the components of the system.

1.2.5.3 Lagrange-Euler Method

$$L(q, \dot{q}) = T - U = E_c - E_p = Constant$$

where L: is the Lagrange of the system.

In the case of a conservative systems, the forces derive from a potential.

We define the action of the system as a summation, between the time interval, $[t_0, t_1]$ along the trajectory of the system, of the difference between the kinetic energy and the potential energy.

$$\Gamma = \int_{t_0}^{t_1} L(q, \dot{q}) dt$$

The trajectory is determined by a variational method. This method results in the Euler-Lagrange equations which give trajectories on which the action is minimal.

- By applying the principle of least action, $\partial\Gamma = 0$, we obtain the Euler-Lagrange equation for a conservative system as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \quad i = 1, n$$

- The motion equation for a dissipative (non-conservative) system can be determined as follows:

  - ✓ ***Translational System***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = \overline{\sum \left|\vec{F}_{ext}\right|} \quad i = 1, n$$

where $\vec{F}_{ext}$ are the external forces applied to the system.

  - ✓ ***Rotational System***

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = \overline{\sum \left|\vec{M}_{ext}\right|} \quad i = 1, n$$

where $\vec{M}_{ext}$ are the external moments applied to the system. In this case the forces do not derive from a potential.