Solution of Exercise n°1

1) Augmented Matrix Form $[A \vdots b]$

The system is written in augmented matrix form as follows:

$$[A \vdots b] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 3 & 2 & 5 \\ 3 & -2 & 4 & 10 \end{array}\right]$$

2) Gauss Elimination Method

Step 1: Eliminate elements below the pivot $a_{11} = 2$.

- $L_2 \leftarrow L_2 - \frac{1}{2}L_1$
- $L_3 \leftarrow L_3 - \frac{3}{2}L_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} \\ 0 & -\frac{7}{2} & \frac{11}{2} & \frac{11}{2} \end{array}\right]$$

Step 2: Eliminate the element below the pivot $a_{22} = \frac{5}{2}$.

- $L_3 \leftarrow L_3 - \left(\frac{-7/2}{5/2}\right) L_2 \implies L_3 \leftarrow L_3 + \frac{7}{5}L_2$

The resulting augmented upper triangular matrix $[\tilde{A} \vdots \tilde{b}]$ is:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} \\ 0 & 0 & 9 & \frac{52}{5} \end{array}\right]$$

3) Back Substitution

Solving the system from the bottom up:

- From L_3: $9x_3 = \frac{52}{5} \implies x_3 = \frac{52}{45}$
- From L_2: $\frac{5}{2}x_2 + \frac{5}{2}(\frac{52}{45}) = \frac{7}{2} \implies 5x_2 = 7 - \frac{52}{9} = \frac{11}{9} \implies x_2 = \frac{11}{45}$
- From L_1: $2x_1 + \frac{11}{45} - \frac{52}{45} = 3 \implies 2x_1 = 3 + \frac{41}{45} = \frac{176}{45} \implies x_1 = \frac{88}{45}$

The solution vector is $X = \begin{bmatrix} \frac{88}{45} & \frac{11}{45} & \frac{52}{45} \end{bmatrix}^T$.

4) Determinant of A

For an upper triangular matrix, the determinant is the product of the diagonal elements:

$$\det(A) = 2 \times \frac{5}{2} \times 9 = 45$$

Solution of Exercise n°2

The determinant to evaluate is the Vandermonde determinant of order 3:

$$\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}$$

Step 1: Row Operations

Perform the following operations to create zeros in the first column:

- $L_2 \leftarrow L_2 - L_1$
- $L_3 \leftarrow L_3 - L_1$

$$\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 \\ 0 & x_2 - x_0 & x_2^2 - x_0^2 \end{vmatrix}$$

Step 2: Further Elimination

To obtain an upper triangular matrix, perform: $L_3 \leftarrow L_3 - \frac{x_2 - x_0}{x_1 - x_0} L_2$. The pivot element a_{33} becomes:

$$a_{33} = (x_2^2 - x_0^2) - \frac{x_2 - x_0}{x_1 - x_0}(x_1^2 - x_0^2) = (x_2 - x_0)(x_2 - x_1)$$

The triangular form is:

$$\Delta = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 \\ 0 & 0 & (x_2 - x_0)(x_2 - x_1) \end{vmatrix}$$

Step 3: Final Result

The determinant is the product of the diagonal elements:

$$\Delta = 1 \cdot (x_1 - x_0) \cdot (x_2 - x_0)(x_2 - x_1)$$

$$\Delta = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)$$

Exercise 3: LU Decomposition Method

(a) Matrix Form $AX = b$

The system can be written as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Where:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 5 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

(b) Existence and Uniqueness of LU-decomposition

A matrix A admits a unique LU-decomposition (where $L_{ii} = 1$) if all its leading principal minors are non-zero:

- $detA_[1] = |1| = 1 \neq 0$
- $detA_[2] = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = (1)(3) - (1)(2) = 1 \neq 0$
- $detA_[3] = \det(A) = 1(27-25)-1(18-20)+1(10-12) = 2+2-2 = 2 \neq 0$

Since all leading principal minors are non-zero, the decomposition $A = LU$ exists and is unique.

(c) Determination of L and U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The product LU is given by:

$$LU = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

By identifying the elements of LU with the elements of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 5 & 9 \end{bmatrix}$, we solve row by row:

1. **From the first row:**

$$u_{11} = 1, \quad u_{12} = 1, \quad u_{13} = 1$$

2. **From the second row:**

   - $l_{21}u_{11} = 2 \implies l_{21}(1) = 2 \implies l_{21} = 2$
   - $l_{21}u_{12} + u_{22} = 3 \implies 2(1) + u_{22} = 3 \implies u_{22} = 1$
   - $l_{21}u_{13} + u_{23} = 5 \implies 2(1) + u_{23} = 5 \implies u_{23} = 3$

3. **From the third row:**

   - $l_{31}u_{11} = 4 \implies l_{31}(1) = 4 \implies l_{31} = 4$
   - $l_{31}u_{12} + l_{32}u_{22} = 5 \implies 4(1) + l_{32}(1) = 5 \implies l_{32} = 1$
   - $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 9 \implies 4(1) + 1(3) + u_{33} = 9 \implies u_{33} = 2$

Final Result:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Solving the system $AX = b$

1. **Solve $LY = b$ (Forward substitution):**

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \implies \begin{cases} y_1 = 1 \\ 2(1) + y_2 = 3 \implies y_2 = 1 \\ 4(1) + 1(1) + y_3 = 6 \implies y_3 = 1 \end{cases}$$

2. **Solve $UX = Y$ (Backward substitution):**

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \begin{cases} 2x_3 = 1 \implies x_3 = 1/2 \\ x_2 + 3(1/2) = 1 \implies x_2 = -1/2 \\ x_1 + (-1/2) + 1/2 = 1 \implies x_1 = 1 \end{cases}$$

The solution is $X = \begin{bmatrix} 1 & -1/2 & 1/2 \end{bmatrix}^T$.

(d) Determinant of A

Using the LU decomposition:

$$\det(A) = \det(L) \times \det(U)$$

Since $\det(L) = 1$ (product of unit diagonal):

$$\det(A) = 1 \times (u_{11} \times u_{22} \times u_{33}) = 1 \times 1 \times 2 = 2$$

Exercise 4: Cholesky Factorization Method

(i) Matrix Form $AX = b$

The system (S) can be written in matrix form as:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 12 \\ -3 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 29 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} -13 \\ 12 \\ -3 \end{bmatrix}$.

(ii) Symmetry and Positive Definiteness

- **Symmetry:** A matrix is symmetric if $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 29 \end{bmatrix} = A$.

- **Positive Definite:** Using Sylvester's Criterion (leading principal minors must be positive):

  1. $detA_[1] = |1| = 1 > 0$
  2. $detA_[2] = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = (1)(5) - (2)(2) = 1 > 0$
  3. $detA_[3] = \det(A) = 1(145 - 100) - 2(58 - 30) + 3(20 - 15) = 45 - 56 + 15 = 4 > 0$

  Since all leading principal minors are positive, A is positive definite.

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(iii) Cholesky Factorization $A = MM^T$

Let M be a lower triangular matrix:

$$M = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

The product MM^T is given by:

$$\begin{pmatrix} m_{11}^2 & m_{11}m_{21} & m_{11}m_{31} \\ m_{21}m_{11} & m_{21}^2 + m_{22}^2 & m_{21}m_{31} + m_{22}m_{32} \\ m_{31}m_{11} & m_{31}m_{21} + m_{32}m_{22} & m_{31}^2 + m_{32}^2 + m_{33}^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 29 \end{pmatrix}$$

By identification (assuming $m_{ii} > 0$): * $m_{11}^2 = 1 \implies m_{11} = 1$
* $m_{11}m_{21} = 2 \implies (1)m_{21} = 2 \implies m_{21} = 2$
* $m_{11}m_{31} = 3 \implies (1)m_{31} = 3 \implies m_{31} = 3$
* $m_{21}^2 + m_{22}^2 = 5 \implies 4 + m_{22}^2 = 5 \implies m_{22} = 1$
* $m_{21}m_{31} + m_{22}m_{32} = 10 \implies (2)(3) + (1)m_{32} = 10 \implies m_{32} = 4$
* $m_{31}^2 + m_{32}^2 + m_{33}^2 = 29 \implies 9 + 16 + m_{33}^2 = 29 \implies m_{33} = 2$

Thus:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 2 \end{pmatrix}$$

$$M^T = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

(iv) Solving the system (S)

Step 1: Forward Substitution ($MY = b$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -13 \\ 12 \\ -3 \end{pmatrix} \implies \begin{cases} y_1 = -13 \\ y_2 = 12 - 2(-13) = 38 \\ y_3 = \frac{-3-3(-13)-4(38)}{2} = -58 \end{cases}$$

Step 2: Backward Substitution ($M^T X = Y$)

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -13 \\ 38 \\ -58 \end{pmatrix} \implies \begin{cases} x_3 = -29 \\ x_2 = 38 - 4(-29) = 154 \\ x_1 = -13 - 2(154) - 3(-29) = -234 \end{cases}$$

(v) Determinant of A

Using the Cholesky factor:

$$\det(A) = (\det(M))^2 = (m_{11} \cdot m_{22} \cdot m_{33})^2 = (1 \cdot 1 \cdot 2)^2 = 4$$

Exercise n°5: Solution

(i) Unique LU Decomposition

The matrix A admits a unique LU decomposition (with $l_{ii} = 1$) if all its leading principal minors Δ_i are non-zero:

- $detA_[1] = -2 \neq 0$
- $detA_[2] = (-2)(5) - (-4)(1) = -6 \neq 0$

- $detA_[3] = -2(-5+6) - 1(4-0) = -6 \neq 0$
- $detA_[4] = \det(A) = -22 \neq 0$

Since all $\Delta_i \neq 0$, the decomposition exists and is unique.

(ii) Determination of L and U

Following the property of tridiagonal matrices where $l_{ii} = 1$, the super-diagonal of U is identical to the super-diagonal of A. We define:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & 1 & 0 & 0 \\ 0 & u_{22} & 2 & 0 \\ 0 & 0 & u_{33} & -1 \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

The product LU is:

$$LU = \begin{bmatrix} u_{11} & 1 & 0 & 0 \\ l_{21}u_{11} & l_{21}(1) + u_{22} & 2 & 0 \\ 0 & l_{32}u_{22} & l_{32}(2) + u_{33} & -1 \\ 0 & 0 & l_{43}u_{33} & l_{43}(-1) + u_{44} \end{bmatrix}$$

By identification with matrix A:

1. $u_{11} = -2$
2. $l_{21}u_{11} = -4 \implies l_{21} = 2$
3. $l_{21}(1) + u_{22} = 5 \implies 2 + u_{22} = 5 \implies u_{22} = 3$
4. $l_{32}u_{22} = -3 \implies l_{32} = -1$
5. $l_{32}(2) + u_{33} = -1 \implies -2 + u_{33} = -1 \implies u_{33} = 1$
6. $l_{43}u_{33} = -1 \implies l_{43} = -1$
7. $l_{43}(-1) + u_{44} = 4 \implies 1 + u_{44} = 4 \implies u_{44} = 3$

Thus:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(iii) Solving $AX = b$

We solve $LY = b$ then $UX = Y$ with $b = [-4, 36, -27, 12]^T$.

Step 1: Forward Substitution ($LY = b$)

$$\begin{cases} y_1 = -4 \\ 2y_1 + y_2 = 36 \implies y_2 = 44 \\ -y_2 + y_3 = -27 \implies y_3 = 17 \\ -y_3 + y_4 = 12 \implies y_4 = 29 \end{cases} \implies Y = \begin{bmatrix} -4 \\ 44 \\ 17 \\ 29 \end{bmatrix}$$

Step 2: Backward Substitution ($UX = Y$)

$$\begin{cases} 3x_4 = 29 \implies x_4 = \frac{29}{3} \\ x_3 - x_4 = 17 \implies x_3 = \frac{80}{3} \\ 3x_2 + 2x_3 = 44 \implies 3x_2 = 44 - \frac{160}{3} \implies x_2 = -\frac{28}{9} \\ -2x_1 + x_2 = -4 \implies -2x_1 = -4 + \frac{28}{9} \implies x_1 = \frac{4}{9} \end{cases}$$

Final Solution:

$$X = \begin{bmatrix} 4/9 \\ -28/9 \\ 80/3 \\ 29/3 \end{bmatrix}$$