Mohamed Boudiaf University - M'sila
Faculty of Technology
Department of Civil Engineering
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Module: Numerical Methods

Solutions - Tutorial Series No. 4

Iterative Methods for Solving Linear Systems

Exercise 1: Reordering for Strict Diagonal Dominance

$$\begin{cases} 2x_1 + x_2 + 4x_3 = 7 \\ 5x_1 + 2x_2 - x_3 = 6 \\ x_1 + 6x_2 - 3x_3 = 4 \end{cases}$$

1. Reorder the equations so that the matrix becomes strictly diagonally dominant.
2. Using the Jacobi and Gauss-Seidel methods, compute the first 3 iterations starting from $X^{(0)} = (0,0,0)^T$.

1) Reordering
We reorder the system as follows:

$$\begin{cases} 5x_1 + 2x_2 - x_3 = 6 \\ x_1 + 6x_2 - 3x_3 = 4 \\ 2x_1 + x_2 + 4x_3 = 7 \end{cases}$$

Check diagonal dominance:

$$|5| > 2 + 1, \quad |6| > 1 + 3, \quad |4| > 2 + 1$$

Hence, the system is strictly diagonally dominant.

2) Iterative formulas

$$x_1 = \frac{6 - 2x_2 + x_3}{5}, \quad x_2 = \frac{4 - x_1 + 3x_3}{6}, \quad x_3 = \frac{7 - 2x_1 - x_2}{4}$$

Jacobi Method

$$X^{(0)} = (0, 0, 0)$$

Iteration 1

$$X^{(1)} = (1.200,\ 0.667,\ 1.750)$$

Iteration 2

$$X^{(2)} = (1.283,\ 1.342,\ 0.983)$$

Iteration 3

$$X^{(3)} = (0.860,\ 0.944,\ 0.773)$$

Gauss-Seidel Method

Iteration 1

$$X^{(1)} = (1.200,\ 0.467,\ 1.033)$$

Iteration 2

$$X^{(2)} = (1.220,\ 0.980,\ 0.895)$$

Iteration 3

$$X^{(3)} = (0.987,\ 0.950,\ 1.019)$$

Remark
Gauss-Seidel converges faster than Jacobi. The solution is close to:

$$(1, 1, 1)$$

Exercise 2: Matrices and Convergence Criteria

a) Decomposition $A = D - E - F$:

$$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

b) Jacobi iteration matrix (J):

$$J = D^{-1}(E + F) = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.2 & 0.2 \\ -0.1 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix}$$

c) Gauss-Seidel iteration matrix (G): First, we find $(D - E)^{-1}$ using forward substitution:

$$(D - E)^{-1} = \begin{pmatrix} 10 & 0 & 0 \\ 1 & 10 & 0 \\ 2 & 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 0.1 & 0 & 0 \\ -0.01 & 0.1 & 0 \\ -0.018 & -0.02 & 0.1 \end{pmatrix}$$

$$G = (D - E)^{-1}F = \begin{pmatrix} 0.1 & 0 & 0 \\ -0.01 & 0.1 & 0 \\ -0.018 & -0.02 & 0.1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.2 & 0.2 \\ 0 & 0.02 & -0.12 \\ 0 & 0.036 & -0.016 \end{pmatrix}$$

d) Infinity norms:

- $\|J\|_\infty = \max(|-0.2| + |0.2|,\ |-0.1| + |-0.1|,\ |-0.2| + |-0.2|) = \max(0.4, 0.2, 0.4) = \mathbf{0.4}$
- $\|G\|_\infty = \max(|-0.2|+|0.2|,\ |0.02|+|-0.12|,\ |0.036|+|-0.016|) = \max(0.4, 0.14, 0.052) = \mathbf{0.4}$

e) Convergence Conclusion: Since $\|J\|_\infty = 0.4 < 1$ and $\|G\|_\infty = 0.4 < 1$, both methods are strictly guaranteed to converge. Although both have the same infinity norm, Gauss-Seidel suppresses the error much faster in the lower rows, making it practically faster.

Exercise 3: Convergence Study with Norms

1. Verify strict diagonal dominance:
Row 1: $|6| > |-1| + |1| \implies 6 > 2$ (True).
Row 2: $|7| > |-2| + |-1| \implies 7 > 3$ (True).
Row 3: $|8| > |1| + |-1| \implies 8 > 2$ (True).
Condition holds.

2. Derive the Jacobi iteration matrix J:

$$J = D^{-1}(E+F) = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{7} & 0 & \frac{1}{7} \\ -\frac{1}{8} & \frac{1}{8} & 0 \end{pmatrix}$$

3. Compute infinity norm $\|J\|_\infty$:
$\|J\|_\infty = \max\left(\frac{2}{6}, \frac{3}{7}, \frac{2}{8}\right) = \max(0.333, 0.428, 0.250) = \frac{3}{7}$.

4. Conclusion:
Yes, $\|J\|_\infty = \frac{3}{7} < 1$ strictly holds. This proves that a strictly diagonally dominant matrix always yields an iteration matrix whose norm is less than 1, unconditionally guaranteeing convergence.

Exercise 4: Successive Over-Relaxation (SOR) Method

(a) Diagonal dominance

We check:

$$|10| > |2| + |-1| = 3$$

$$|-6| > |-3| + |2| = 5$$

$$|5| > |1| + |1| = 2$$

Hence, the matrix is **strictly diagonally dominant**, ensuring convergence.

(b) First iteration $x^{(1)}$
Using $\omega = 1.1$ and $x^{(0)} = 0$:

$$x_1^{(1)} = (1 - 1.1) \cdot 0 + \frac{1.1}{10}(27) = 2.97$$

$$x_2^{(1)} = \frac{1.1}{-6}(-61.5 + 3 \cdot 2.97) = \frac{1.1}{-6}(-52.59) \approx 9.64$$

$$x_3^{(1)} = \frac{1.1}{5}(-21.5 - 2.97 - 9.64) = \frac{1.1}{5}(-34.11) \approx -7.50$$

Thus:

$$x^{(1)} \approx (2.97,\ 9.64,\ -7.50)$$

Second iteration $x^{(2)}$

$$x_1^{(2)} = (1 - 1.1)(2.97) + \frac{1.1}{10}(27 - 2 \cdot 9.64 + 7.50) \approx 1.74$$

$$x_2^{(2)} = (1 - 1.1)(9.64) + \frac{1.1}{-6}(-61.5 + 3 \cdot 1.74 - 2(-7.50)) \approx 7.12$$

$$x_3^{(2)} = (1 - 1.1)(-7.50) + \frac{1.1}{5}(-21.5 - 1.74 - 7.12) \approx -6.31$$

$$x^{(2)} \approx (1.74,\ 7.12,\ -6.31)$$

(c) Discussion

- $\omega > 1$ accelerates convergence (over-relaxation).
- $\omega = 1$ reduces to the Gauss-Seidel method.
- $\omega \geq 2$ typically leads to divergence.