Mohamed Boudiaf University - M'sila **University Year: 2025/2026**
Faculty of Technology
Department of Civil Engineering
Module: Numerical Methods

Solutions to the Directed Works Series n°1

Exercise n° 1:
All digits in the following numbers are **correct**. Calculate the absolute and relative errors:
a) 0.1342 *b*) 7.480 *c*) 0.34235×10^{-3} *d*) 7.000

Solution:

Since all digits are correct, the absolute error is $\Delta x = 0.5 \times 10^k$ where k is the last decimal rank.

a) $x = 0.1342$: Last rank $10^{-4} \implies \Delta x = 0.5 \times 10^{-4} = \mathbf{0.00005}$.
$r_x = \frac{0.00005}{0.1342} \approx \mathbf{3.72 \times 10^{-4}}$.

b) $x = 7.480$: Last rank $10^{-3} \implies \Delta x = 0.5 \times 10^{-3} = \mathbf{0.0005}$.
$r_x = \frac{0.0005}{7.480} \approx \mathbf{6.68 \times 10^{-5}}$.

c) $x = 0.34235 \times 10^{-3}$: Last rank $10^{-8} \implies \Delta x = 0.5 \times 10^{-8}$.
$r_x = \frac{0.5\times 10^{-8}}{0.34235\times 10^{-3}} \approx \mathbf{1.46 \times 10^{-5}}$.

d) $x = 7.000$: Last rank $10^{-3} \implies \Delta x = \mathbf{0.0005}$, $r_x \approx \mathbf{7.14 \times 10^{-5}}$.

Exercise n° 2:
Let $x = 0.1256$ be given with an error of 0.5%..
1) Give the absolute error of x.
2) Determine the exact number of significant digits of this number.
3) Round the result to the last csd .

Solution:
1) The absolute error of x:
The relationship between absolute error Δx and relative error δ_x is given by:

$$\Delta x = |x| \cdot r_x$$

Given $x = 0.1256$ and $r_x = 0.005$ (which is 0.5%), we have:

$$\Delta x = 0.1256 \times 0.005 = 0.000628$$

So, the absolute error is: $\Delta x = 0.0628 \times 10^{-2}$.
2) Exact number of correct significant digits (csd):
(m=-1)

$$\Delta x = 0.0628 \times 10^{-2} \leq \times 0.5 \times 10^{-2} \leq 0.5 \times 10^{m-n+1} \Rightarrow m - n + 1 = -2 \Rightarrow n = 2$$

Therefore, the number of exact significant digits is **2**.
3) Rounding the result to the last csd:

$$x \approx 0.13$$

Final result $x = 0.13 \pm 0.01$

Exercise n° 3:
Let: $x = 2.5 \pm 0.01$; $y = 1.2 \pm 0.02$; $z = 3.2 \pm 0.03$; $t = 5.1 \pm 0.01$.
1) Verify that all digits of x, y, z, and t are **correct**.
2) Let $S = x^2 + y^2 + z^2 + t^2$.
a) Determine the number of **CSD** of S.
b) Round the result to the last **CSD** and give the final result.

Solution:

1. **Verification (for x):** $\Delta x = 0.01$, $m = 0$.
$0.01 = 0.1 \times 10^{-1} \leq 0.5 \times 10^{-1} \leq 0.5 \times 10^{0-n+1} \Rightarrow n = 2$ (True). All have $n = 2$ CSD.

2. **Calculation of S:** $S^* = 2.5^2 + 1.2^2 + 3.2^2 + 5.1^2 = \mathbf{43.94}$.
   Error ΔS: $\Delta S = \sum |2x_i| \Delta x_i = 2(2.5 \cdot 0.01 + 1.2 \cdot 0.02 + 3.2 \cdot 0.03 + 5.1 \cdot 0.01) = \mathbf{0.392} \simeq 0.4$.

3. **CSD for S:** $m = 1$. For $n = 2$: $\Delta S \leq 0.5 \times 10^{1-2+1} = 0.5$ (True). $\mathbf{n = 2}$.

4. **Final result:** $\mathbf{S \approx 44 \pm 0.8}$.

***Exercise n° 4:**

Let: $x = 5.27 \pm 0.01$; $y = 28.61 \pm 0.04$; $z = 15.80 \pm 0.02$.

Calculate for $f = \dfrac{x^2 y^{1/3}}{z^3}$:

a) The value of f and its relative and absolute errors.

b) Give the final result of f in the form $f^* \pm \Delta f$.

Solution:

$f = \frac{x^2 y^{1/3}}{z^3} \implies f^* = \frac{5.27^2 \cdot 28.61^{1/3}}{15.80^3} \approx \mathbf{0.0215}$.

1. **Relative error:** $r_f = 2r_x + \frac{1}{3}r_y + 3r_z$
   $r_f = 2(\frac{0.01}{5.27}) + \frac{1}{3}(\frac{0.04}{28.61}) + 3(\frac{0.02}{15.80}) \approx 0.003795 + 0.000466 + 0.003797 \approx \mathbf{0.008058} \simeq 0.008$.

2. **Absolute error:** $\Delta f = f^* \cdot r_f \approx 0.0215 \times 0.008 \approx \mathbf{0.0002}$.

3. **Final form:** $\mathbf{f = 0.0215 \pm 0.0002}$.

Exercise n ∘ 5:

Consider the equation $f(x) = x^3 - x - 3 = 0$.

1) Graphically find an interval [a, b] of length 1 containing the root α.

2) Using the bisection method with precision $\varepsilon = 10^{-2}$:

a) Calculate the number of iterations needed.

b) Find the root α.

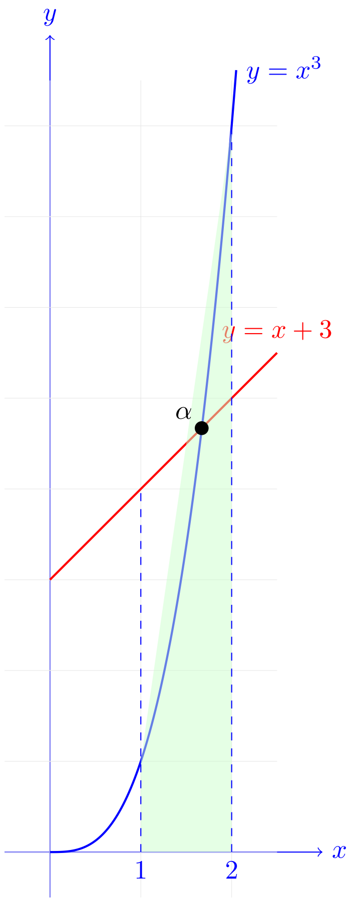
Solution:

1) Graphical Isolation of the Root α

To determine an interval $[a, b]$ of length 1, we rewrite the equation as follows:

$$x^3 - x - 3 = 0 \iff \mathbf{x^3 = x + 3}$$

Let $g_1(x) = x^3$ and $g_2(x) = x + 3$. The root α is the x-coordinate of the intersection point.



The intersection occurs within $[1, 2]$ because the curves switch positions. Both functions are strictly increasing, ensuring a **unique root** $\alpha \in [1, 2]$.

2) Bisection Method with Precision $\varepsilon = 10^{-2}$

a) Number of Iterations (n)

Using the refined error formula for the midpoint c_n:

$$n \geq \frac{\ln\left(\frac{2-1}{2\times 10^{-2}}\right)}{\ln 2} \approx 5.64 \implies \mathbf{n = 6}$$

b) Calculation Table

k	a_k	b_k	$x_k = \frac{a_k+b_k}{2}$	$Sign f(x_k)$	$\frac{b_k - a_k}{2}$
0	1.0	2.0	1.5	$(-)$	0.5
1	1.5	2.0	1.75	$(+)$	0.25
2	1.5	1.75	1.625	$(-)$	0.125
3	1.625	1.75	1.6875	$(+)$	0.0625
4	1.625	1.6875	1.65625	$(-)$	0.03125
5	1.65625	1.6875	1.671875	$(+)$	0.015625
6	1.65625	1.671875	**1.6640625**	$(-)$	0.0078125

After 7 steps (from $k = 0$ to $k = 6$), the root approximation is:

Numerical Root: $\alpha \approx \mathbf{1.6640625}$

Or, the error $|1.6640625 - \alpha| \leq \varepsilon$ confirms the precision $\varepsilon = 10^{-2}$.

Exercise n ∘ 6:

Calculate $\sqrt{3}$ with a precision of $\varepsilon = 10^{-2}$ using the bisection method (Hint: use $** f(x) = x^2 - 3$).

Solution:

1) Root Isolation

We evaluate the function $x = \sqrt{3} \Leftrightarrow f(x) = x^2 - 3 = 0$ to find an interval $[a, b]$ of length 1:

- $f(1) = 1^2 - 3 = -2 < 0$
- $f(2) = 2^2 - 3 = 1 > 0$

Since $f(1) \cdot f(2) < 0$ and f is continuous, the root $\alpha = \sqrt{3}$ lies in $[1, 2]$. **Monotonicity:** $f'(x) = 2x > 0$ on $[1, 2]$, ensuring the root is unique.

3) Bisection Table

k	a_k	b_k	$x_k = \frac{a_k+b_k}{2}$	$Sign f(x_k)$	$\frac{b_k - a_k}{2}$
0	1.0	2.0	1.5	$(-)$	0.5
1	1.5	2.0	1.75	$(+)$	0.25
2	1.5	1.75	1.625	$(-)$	0.125
3	1.625	1.75	1.6875	$(-)$	0.0625
4	1.6875	1.75	1.71875	$(-)$	0.03125
5	1.71875	1.75	1.734375	$(+)$	0.015625
6	1.71875	1.734375	**1.7265625**	$(-)$	**0.0078125**

4) Final Result

the error $\dfrac{|1.71875 - 1.734375|}{2} \leq \varepsilon$, the approximation of $\sqrt{3}$ is:

$$\boxed{\textbf{Approximate Value: } \alpha \approx \mathbf{1.7265625}}$$

<u>Exercise n ∘ 7:</u> Consider $f(x) = \exp(x) - x - 2 = 0$.

1. Show that $f(x) = 0$ has a unique solution $\alpha \in [1, 2]$.
2. Consider the fixed-point forms: $g_1(x) = e^x - 2$ and $g_2(x) = \ln(x + 2)$.
   (a) Which function satisfies the fixed-point conditions? Justify.
   (b) Starting with $x_0 = 1.5$, calculate the number of iterations for $\varepsilon = 10^{-3}$.
   (c) Calculate the root α and show the error $|x_1 - x_0|$.

Solution

1) Existence and Uniqueness of α

- **Existence:** $f(x) = e^x - x - 2$ is continuous on $I = [1, 2]$.
  - $f(1) = e^1 - 1 - 2 \approx -0.2817 < 0$
  - $f(2) = e^2 - 2 - 2 \approx 3.3891 > 0$

  Since $f(1) \cdot f(2) < 0$, by the *Intermediate Value Theorem*, there exists at least one root $\alpha \in [1, 2]$.
- **Uniqueness:** $f'(x) = e^x - 1$. For $x \in [1, 2]$, $f'(x) \geq e^1 - 1 \approx 1.718 > 0$. Since the derivative is strictly positive, the function is strictly increasing, which ensures the **uniqueness** of α.

We evaluate the two conditions: **Stability** $(g(I) \subseteq I)$ and **Contraction** $(|g'(x)| \leq M < 1)$.

Analysis of $g_1(x) = e^x - 2$:

1. **Stability Check:** for $x \in [1, 2] : 1 \leq x \leq 2 \Leftrightarrow e^1 - 2 \leq e^x - 2 \leq e^2 - 2 \Leftrightarrow 0.718 \leq g_1(x) \leq 3.3891$ Since , $g_1(I) \not\subseteq I$. The **Stability condition fails**.
2. **Contraction Check:** $g_1'(x) = e^x$. For $x \in]1, 2[$, $|g_1'(x)| > e^1 \approx 2.718 > 1$. The **Contraction condition fails**.

$\Longrightarrow$ $g_1(x)$ **is divergent** and cannot be used.

Analysis of $g_2(x) = \ln(x+2)$:

1. **Stability Check:** for $x \in [1,2] : 1 \leq x \leq 2 \Leftrightarrow 3 \leq x+2 \leq 4 \Leftrightarrow 1.098 \leq g_2(x) \leq 1.386$
   Since $[1.098, 1.386] \subseteq [1,2]$, the **Stability condition is satisfied**.
2. **Contraction Check:** $g_2'(x) = \frac{1}{x+2}$. On $]1,2[$, the maximum value is: $M = |g_2'(1)| = \frac{1}{3} \approx 0.333 < 1$. The **Contraction condition is satisfied**.

$\implies$ $g_2(x)$ **is convergent** and satisfies the Fixed-Point Theorem.

2-b) Iteration Count and Initial Error

With $x_0 = 1.5$ and $M = 1/3$:

- $x_1 = \ln(1.5+2) = \ln(3.5) \approx 1.25276$
- **Initial Error:** $|x_1 - x_0| = |1.25276 - 1.5| = 0.24724$

Using $|x_n - \alpha| \leq \frac{M^n}{1-M}|x_1 - x_0| \leq 10^{-3}$:

$$\frac{(1/3)^n}{1-1/3}(0.24724) \leq 10^{-3} \implies (1/3)^n \leq \frac{10^{-3} \cdot 2/3}{0.24724} \approx 0.002696$$

$$n \geq \frac{\ln(0.002696)}{\ln(1/3)} \approx 5.38 \implies \mathbf{n = 6} \text{ iterations}$$

2-c) Root Calculation Table

n	x_n	Error $\|x_n - x_{n-1}\|$
0	1.50000	-
1	1.25276	0.24724
2	1.17950	0.07326
3	1.15672	0.02278
4	1.14954	0.00718
5	1.14726	0.00228
6	1.14654	0.00072

Table 1: Fixed-point iterations for $g_2(x)$

The approximate root is $\alpha \approx 1.14654$.

Exercise 8: Let $f(x) = x\ln x - 3 = 0$.

1. Show that the root lies in $[2,3]$.
2. Verify the convergence conditions for the Newton-Raphson method in $[2,3]$.
3. Calculate the root with $\varepsilon = 10^{-3}$, starting from $x_0 = 3$.

1) Existence of the Root

We apply the **Intermediate Value Theorem (IVT)** on the interval $[2,3]$:

- $f(2) = 2\ln(2) - 3 \approx 1.3863 - 3 = -1.6137 < 0$
- $f(3) = 3\ln(3) - 3 \approx 3.2958 - 3 = +0.2958 > 0$

Since $f(x)$ is continuous on $[2,3]$ and $f(2) \cdot f(3) < 0$, there exists at least one root $\alpha \in (2,3)$.

2) Verification of Convergence Conditions

To guarantee convergence, we check the following conditions on $[2, 3]$:

- **Differentiability:** $f(x)$ is C^2 on $[2, 3]$.
  $f'(x) = \ln x + 1$ and $f''(x) = \frac{1}{x}$.
- **Monotonicity ($f'(x) \neq 0$):** For $x \in [2, 3]$, $f'(x) > 0$. The function is strictly increasing, thus the root is unique.
- **Constant Concavity ($f''(x) \neq 0$):** For $x \in [2, 3]$, $f''(x) > 0$. The curve is strictly convex (no inflection points).
- **Initial Guess Choice (x_0):** We choose x_0 such that $f(x_0) \cdot f''(x_0) > 0$. Since $f''(x) > 0$, we select $x_0 = 3$ (where $f(3) > 0$).

3) Numerical Iterations

The iteration formula is: $x_{n+1} = x_n - \dfrac{x_n \ln x_n - 3}{\ln x_n + 1}$

n	x_n	$f(x_n)$	$f'(x_n)$	$\lvert x_{n+1} - x_n \rvert$
0	3.000000	0.295837	2.098612	—
1	2.859032	0.003310	2.050484	0.140968
2	2.857417	0.000000	2.049919	0.001615
3	**2.857417**	0.000000	2.049919	**0.000000**

Table 2: Newton-Raphson Iterations ($\varepsilon = 10^{-3}$)

Conclusion: The root is $\alpha \approx \mathbf{2.857417}$ after 3 iterations.