Department of Civil Engineering
Course: Numerical Methods
Quiz - Time: 1 Hour

Test 1

Exercise 1: Thomas Algorithm (5 points)

Consider the following linear system (S):

$$\begin{cases} 5x_1 - 2x_2 = 10 \\ -10x_1 + 8x_2 + x_3 = 26 \\ -4x_2 + x_3 = 6 \end{cases}$$

a) Matrix form (0.75 point)
Write the system in the matrix form $AX = b$.

b) Type of matrix (0.25 point)
Identify the type of the matrix A and justify your answer.

c) LU decomposition (2 points)
Determine the matrices L and U such that

$$A = LU,$$

where L is a lower bidiagonal matrix with ones on the diagonal and U is an upper bidiagonal matrix.

d) Resolution (2 points)
Solve the system using the Thomas algorithm (forward elimination and back substitution).

Exercise 2: Convergence Study (5 points)

$$A = \begin{pmatrix} 5 & -2 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & 4 \end{pmatrix}$$

1. Decompose $A = D - E - F$, where **(1 point)**:
   - D is the diagonal part of A,
   - $-E$ is the strictly lower triangular part of A,
   - $-F$ is the strictly upper triangular part of A.
2. Determine the **Gauss–Seidel iteration matrix (3 points)**:
$$G = (D - E)^{-1}F$$
3. Compute the **infinity norm** of the iteration matrix **(1 point)**:
$$\|G\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |g_{ij}|$$

Correction

Correction of Exercise 1 (5 points)

a) Matrix form (0.75 pt)

$$A = \begin{pmatrix} 5 & -2 & 0 \\ -10 & 8 & 1 \\ 0 & -4 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 26 \\ 6 \end{pmatrix}$$

$$AX = b$$

Points: 0.75

b) Type of matrix (0.25 pt)

Matrix A is tridiagonal.

Justification: Nonzero entries appear only on the main, upper, and lower diagonals.

Points: 0.25

c) LU decomposition (2 pts)

$$A = LU$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_2 & 1 & 0 \\ 0 & \ell_3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_1 & c_1 & 0 \\ 0 & u_2 & c_2 \\ 0 & 0 & u_3 \end{pmatrix}$$

Step 1

$$u_1 = 5, \quad c_1 = -2$$

Step 2

$$\ell_2 = \frac{-10}{5} = -2$$

$$u_2 = 8 - (-2)(-2) = 4$$

$$c_2 = 1$$

Step 3

$$\ell_3 = \frac{-4}{4} = -1$$

$$u_3 = 1 - (-1)(1) = 2$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 5 & -2 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Points: 2

d) Resolution using Thomas algorithm (2 pts)

Step 1: Forward substitution ($LY = b$)

$$y_1 = 10$$

$$-2y_1 + y_2 = 26 \Rightarrow y_2 = 26 + 20 = 46$$

$$-y_2 + y_3 = 6 \Rightarrow y_3 = 6 + 46 = 52$$

$$Y = \begin{pmatrix} 10 \\ 46 \\ 52 \end{pmatrix}$$

Step 2: Back substitution ($UX = Y$)

$$2x_3 = 52 \Rightarrow x_3 = 26$$

$$4x_2 + x_3 = 46 \Rightarrow 4x_2 + 26 = 46 \Rightarrow x_2 = 5$$

$$5x_1 - 2x_2 = 10 \Rightarrow 5x_1 - 10 = 10 \Rightarrow x_1 = 4$$

$$X = \begin{pmatrix} 4 \\ 5 \\ 26 \end{pmatrix}$$

Points: 2

Final Answer: $(x_1, x_2, x_3) = (4, 5, 26)$

Total: 5 / 5

Correction of Exercise 2 (5 points)

$$A = \begin{pmatrix} 5 & -2 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & 4 \end{pmatrix}$$

1) Decomposition $A = D - E - F$ (1 pt)

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix} \quad (\text{since } -E \text{ is the lower part of } A)$$

$$F = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{since } -F \text{ is the upper part of } A)$$

Points: 1

2) Gauss–Seidel matrix $G = (D - E)^{-1}F$ (3 pts)

$$D - E = \begin{pmatrix} 5 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

Inverse of $(D - E)$:

$$(D - E)^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{20} & \frac{1}{4} & 0 \\ -\frac{9}{80} & -\frac{1}{16} & \frac{1}{4} \end{pmatrix}$$

Compute $G = (D - E)^{-1}F$:

$$G = \begin{pmatrix} 0 & \frac{2}{5} & -\frac{2}{5} \\ 0 & \frac{1}{10} & -\frac{7}{20} \\ 0 & -\frac{9}{40} & \frac{23}{80} \end{pmatrix}$$

Points: 3

3) Infinity norm $\|G\|_\infty$ (1 pt)

$$\|G\|_\infty = \max(\text{sum of absolute values of rows})$$

$$\text{Row 1: } \frac{2}{5} + \frac{2}{5} = \frac{4}{5}$$

$$\text{Row 2: } \frac{1}{10} + \frac{7}{20} = \frac{9}{20}$$

$$\text{Row 3: } \frac{9}{40} + \frac{23}{80} = \frac{41}{80}$$

$$\|G\|_\infty = \frac{4}{5}$$

Points: 1

Final Result: $\|G\|_\infty = \frac{4}{5} < 1$

The Gauss–Seidel method converges.

Total: 5 / 5