Mohamed Boudiaf University - M'sila **University Year : 2024/2025**
Faculty of Technology
Department of Civil Engineerin
Module : Numerical Methods

Sample solution the resit exam

Exercice n°1 : (7 marks)

Consider the function $f(x)$ defined by :$f(x) = x^3 - x - 1$ on the interval $[1; 2]$.

1. Show that $f(x) = 0$ has a unique solution α in the interval $[1; 2]$.
2. Verify that the function $g(x) = \sqrt[3]{x+1}$ satisfies the conditions of the fixed point theorem in the interval $[1; 2]$.
3. Calculate the root α with $x_0 = 1.5$ and the precision $\epsilon = 10^{-3}$ (take 4 digits after the decimal point).

Solution :

1. The function f is continuous on R and so on the interval $[1; 2]$ (0.25 mark)
   $f(1) \times f(2) < 0$ (0.25 mark)
   $f'(x) = 3x^2 - 1 > 0, \forall x \in [1; 2]$ (0.25 mark)
   So, according to the intermediate value theorem, there exists a unique $\alpha \in [1; 2]$. (0.25 mark)
2. **i)** $f(x) = 0 \Leftrightarrow x^3 - x - 1 = 0 \Leftrightarrow x^3 = x + 1 \Leftrightarrow x = \sqrt[3]{x+1}$
   $\Leftrightarrow g(x) = \sqrt[3]{x+1}$ (1 mark)

   ii) Let $x \in [1; 2]$, i.e., $1 \leq x \leq 2 \Longleftrightarrow 2 \leq x + 1 \leq 3 \Leftrightarrow \sqrt[3]{2} \leq \sqrt[3]{x+1} \leq \sqrt[3]{3} \Leftrightarrow 1,2599 \leq g(x) \leq 1,4422$, so, $\forall x \in [1; 2] : g(x) \in [1; 2]$. (1 mark)

   iii) $|g'(x)| = |\frac{1}{3(x+1)^{\frac{2}{3}}}| \leq Max|\frac{1}{3(x+1)^{\frac{2}{3}}}| < 0.21 < 1, \forall x \in [1; 2]$ (0.5 mark)
   So, the function $g(x)$ satisfies the conditions of the fixed point theorem in the interval $[1; 2]$, and the sequence

   $$\begin{cases} x_0 \in [1; 2] \\ x_{n+1} = g(x_n) \end{cases}$$

   converges to the solution α.
3. For $x_0 = 1.5$ and $\epsilon = 10^{-3}$, we obtain :

   $$\begin{aligned} x_1 &= g(x_0) = \sqrt[3]{1.5+1} = 1,3572, \quad |x_1 - x_0| 0,1427 > \epsilon \quad (0.75\ mark) \\ x_2 &= g(x_1) = \sqrt[3]{x_1+1} = 1,3308 \quad |x_2 - x_1| = 0,0263 > \epsilon \quad (0.75\ mark) \\ x_3 &= g(x_2) = \sqrt[3]{x_2+1} = 1,3259; \quad |x_3 - x_2| = 0.005 > \epsilon \quad (0.75\ mark) \\ x_4 &= g(x_3) = \sqrt[3]{x_3+1} = 1,3249; \quad |x_4 - x_3| < \epsilon \quad (0.75\ mark) \end{aligned}$$

   So we take $\alpha = 1,3249$ (0.5 mark)

Exercice n°2 (9 marks)

Consider the following linear system (S) :

$$\begin{cases} 5x_1 + 3x_2 + x_3 = 12 \\ 10x_1 + 8x_2 + x_3 = -13 \\ -6x_2 + 5x_3 = 161 \end{cases}$$

a) Write the system as $Ax = b$.

b) Show that the matrix A has a unique decomposition $A = LU$, where L is a lower triangular matrix with one diagonal and U is an upper triangular matrix.

c) Determine the matrices L and U, then solve this system.

d) Deduce the determinant of A.

Solution :

a)

$$(S) \Longleftrightarrow Ax = b \Longleftrightarrow \begin{pmatrix} 5 & 3 & 1 \\ 10 & 8 & 1 \\ 0 & -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -13 \\ 161 \end{pmatrix} \text{ (1 } mark\text{)}$$

b) We need to prove that the determinates of all the sub-matrices are invertible, i.e. $\det A_{[1]} = 5 \neq 0$, $\det A_{[2]} = \det \begin{pmatrix} 5 & 3 \\ 10 & 8 \end{pmatrix} = 10 \neq 0$ and $\det A_{[3]} = \det \begin{pmatrix} 5 & 3 & 1 \\ 10 & 8 & 1 \\ 0 & -6 & 5 \end{pmatrix} = 20 \neq 0$. (1 mark)

Then the matrix A has a unique decomposition $A = LU$, where L is a lower triangular matrix with a unit diagonal and U is a triangular matrix.

c) We need to find the two matrices L and U defined as follows $A = LU$.

We put

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, \text{and } U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$\begin{aligned} A &= LU \Leftrightarrow \\ \begin{pmatrix} 5 & 3 & 1 \\ 10 & 8 & 1 \\ 0 & -6 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \times \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \\ &= \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{31} + l_{32}u_{23} + u_{33} \end{pmatrix} \end{aligned}$$

By identification, we obtain

$$\begin{cases} u_{11} = 5 \\ u_{12} = 3 \\ u_{13} = 1 \end{cases} \text{and } \begin{cases} l_{21}u_{11} = 10 \Rightarrow l_{21} = 2 \\ l_{31}u_{11} = 0 \Rightarrow l_{31} = 0 \end{cases} \text{and } \begin{cases} l_{21}u_{12} + u_{22} = 8 \Rightarrow u_{22} = 2 \\ l_{21}u_{13} + u_{23} = 1 \Rightarrow u_{23} = -1 \end{cases}$$

$$\text{and } \begin{cases} l_{31}u_{12} + l_{32}u_{22} = -6 \Rightarrow l_{32} = -3 \\ l_{31}u_{31} + l_{32}u_{23} + u_{33} = 5 \Rightarrow u_{33} = 2 \end{cases}$$

Finally :

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \text{ (1.5 } mark\text{) and } U = \begin{pmatrix} 5 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \text{ (2 } mark\text{)}$$

The resolution of the system $AX = b$ becomes

$$Ax = b \Leftrightarrow L\,(Ux) = b \Leftrightarrow \left\{ \begin{array}{ll} Ly = b & (1) \\ Lx = y & (2) \end{array} \right. \textcolor{red}{(0.5\ mark)}$$

$$Ly = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -13 \\ 161 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} y_1 = 12 \\ 2y_1 + y_2 = -13 \\ -3y_2 + y_3 = 161 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y_1 = 12 \\ y_2 = -37 \textcolor{red}{(1\ mark)} \\ y_3 = 50 \end{array} \right.$$

$$Ux = y \Leftrightarrow \begin{pmatrix} 5 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -37 \\ 50 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{c} 5x_1 + 3x_2 + x_3 = 12 \\ 2x_2 - x_3 = -37 \\ 2x_3 = 50 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x_1 = 1 \\ x_2 = -6 \textcolor{red}{(1\ mark)} \\ x_3 = 25 \end{array} \right.$$

d)

$$\det(A) = \det(LU) = \det(L) \times \det(U) = \prod_{i=1}^{n} u_{ii} = 5 \times 2 \times 2 = 20. \textcolor{red}{(1\ mark)}$$

Exercice n°3 (4 marks)

Consider the points given in the table below :

x_i	-1	0	1	2
$f(x_i)$	2	1	0	0

a) What is the degree of the polynomial represented by these points ?

b) Find this polynomial using Lagrange's method P.

c) Calculate $P(1, 5)$.

Solution :

a) $n + 1 = 4 \Rightarrow n = 3$ (0.25 mark)

b)

$$\begin{aligned} P_3(x) &= \sum_{i=0}^{3} f(x_i)\ell_i(x) \\ &= f(x_0)l_0\,(x) + f(x_1)l_1\,(x) + f(x_2)l_2\,(x) + f(x_3)l_3\,(x) \\ &= 2l_0\,(x) + l_2\,(x)\ \textcolor{red}{(0.5\ mark)} \end{aligned}$$

$$\begin{aligned} l_0\,(x) &= \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} \times \frac{x - x_3}{x_0 - x_3} \\ &= \frac{x - 0}{-1 - 0} \times \frac{x - 1}{-1 - 1} \times \frac{x - 2}{-1 - 2} \\ &= -\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x \textcolor{red}{(1\ mark)} \end{aligned}$$

$$
\begin{aligned}
l_1(x) &= \frac{x-x_0}{x_1-x_0}\times\frac{x-x_2}{x_1-x_2}\times\frac{x-x_3}{x_1-x_3} \\
&= \frac{x+1}{0+1}\times\frac{x-1}{0-1}\times\frac{x-2}{0-2} \\
&= \frac{1}{2}x^3-x^2-\frac{1}{2}x+1 (1\ mark)
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) &= 2\times(-\frac{1}{6}x^3+\frac{1}{2}x^2-\frac{1}{3}x)+\left(\frac{1}{2}x^3-x^2-\frac{1}{2}x+1\right) \\
&= \frac{1}{6}x^3-\frac{7}{6}x+1 (1\ mark)
\end{aligned}
$$

c) $P(1.5)=\frac{1}{6}\times 1.5^3-\frac{7}{6}\times 1.5+1=-0.1875$ (0.25 mark)

Module manager. Merini.A

Good luck