Mohamed Boudiaf University - M'sila **University Year : 2024/2025**
Faculty of Technology
Department of Civil Engineerin
Module : Numerical Methods

Final Exam Sample Solutions

Theoretical Questions(3 marks)

1. State the Newton–Raphson iterative formula.
2. Define diagonal dominance for the matrix $A = (a_{ij})_{n\times n}$.
3. Give two sufficient conditions for convergence of Jacobi and Gauss–Seidel methods.

Solution :

1. The Newton–Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.(1mark)$$

2. A matrix

$$A = (a_{ij})_{n\times n}$$

   is strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \qquad \forall i \in \{1, 2, ..., n\}.(1mark)$$

3. Two sufficient conditions for convergence of Jacobi and Gauss–Seidel methods are :
   (a) A is strictly diagonally dominant.(0.5 mark)
   (b) $\|J\| < 1$. (0.5 mark)

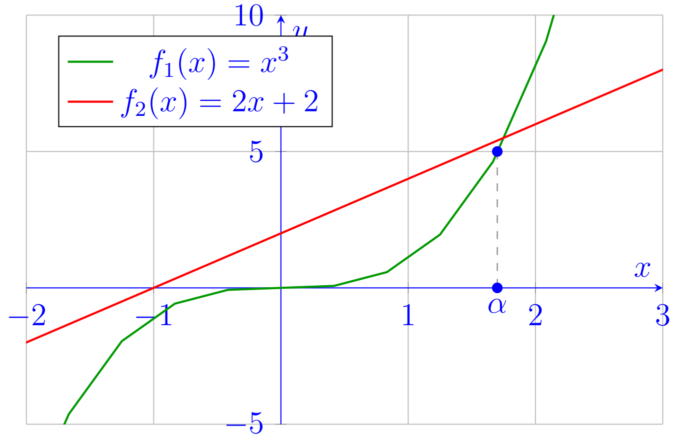
Exercise 1 (5 marks)
Consider the function $f(x) = x^3 - 2x - 2$.

1. Using a graphical argument, propose an interval $[a, b]$ of length 1 containing a root α of $f(x) = 0$, and justify your choice using the Intermediate Value Theorem.
2. Show that Newton–Raphson method can be applied on $[1, 2]$.
3. Approximate the root α using Newton–Raphson method with precision $\varepsilon = 10^{-3}$.

Solution

1. **Graphical localization of the root (0.5 mark)**
   We consider the equation :

$$f(x) = x^3 - 2x - 2 = 0 \quad \Longleftrightarrow \quad x^3 = 2x + 2 \Longleftrightarrow f_1(x) = f_2(x)$$



The function f is continuous on $\mathbb{R}$, hence on $[1, 2]$. (0.25 mark)

$$f(1) = -3, \quad f(2) = 2 \Rightarrow f(1)f(2) < 0.(0.25 mark)$$

Since f is continuous, by the Intermediate Value Theorem
there exists at least one $\alpha \in [1, 2]$ such that :

$$f(\alpha) = 0.(0.25 mark)$$

Moreover,

$$f'(x) = 3x^2 - 2 > 0 \text{ on } [1, 2], (0.25 mark)$$

so the root is unique.

2. **Applicability of Newton–Raphson**
   $f \in C^2([1, 2])$
   i) $f(1)f(2) < 0.$(0.25 mark)
   ii) $f'(x) \neq 0, \forall x \in [1, 2].$ (0.25 mark)
   iii) $f''(x) = 6x > 0, \forall x \in [1, 2].$(0.25 mark)
   Then there exists a unique root $\alpha \in [1, 2]$ of the equation $f(x) = 0$.
   Moreover, if the initial approximation $x_0 = 2$ is chosen such that

$$f(x_0)\, f''(x_0) \geq 0, (0.25 mark)$$

then the sequence defined by the :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}, \quad n \geq 0, (0.5 mark)$$

is well-defined and converges to α.

3. **Newton–Raphson iteration** $\varepsilon = 10^{-3}$, $x_0 = 2$
   Iteration 1 :

$$x_1 = 2 - \frac{2}{10} = 1.8, \quad |x_1 - x_0| = 0.2 > 0. \quad (0.5 mark)$$

Iteration 2 :

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 2}{3x_1^2 - 2} = 1.770, \quad |x_2 - x_1| = 0.03 > 0. \quad (0.5 mark)$$

Iteration 3 :

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 2}{3x_2^2 - 2} = 1.7693, \quad |x_3 - x_2| < 0.001. \quad (0.5 mark)$$

$$x_3 = 1.7693$$

Conclusion :

$$\alpha \approx 1.769 \quad (0.5 mark)$$

<u>Exercise 2</u> (8 marks)
Consider the system :

$$\begin{cases} 7x_1 - 3x_2 + x_3 = -8 \\ -14x_1 + 5x_2 - 3x_3 = 13 \\ 7x_1 - 4x_2 + 4x_3 = -3 \end{cases}$$

Assume that the matrix A admits an LU decomposition without pivoting : $A = LU$, where L is a unit lower triangular matrix and U is upper triangular.

1. Write the system in matrix form $AX = b$.
2. Determine the matrices L and U.
3. Solve the system using LU decomposition.
4. Deduce $\det(A)$.

<u>Solution :</u>

1. Matrix form (AX=b)

$$\begin{cases} 7x_1 - 3x_2 + x_3 = -8 \\ -14x_1 + 5x_2 - 3x_3 = 13 \\ 7x_1 - 4x_2 + 4x_3 = -3 \end{cases} \Leftrightarrow \begin{pmatrix} 7 & -3 & 1 \\ -14 & 5 & -3 \\ 7 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -8 \\ 13 \\ -3 \end{pmatrix} \Leftrightarrow Ax = b(1 \ mark)$$

2. LU decomposition (product method + comparison)

We assume :

$$A = LU,$$

where L is unit lower triangular and U is upper triangular.

Let :

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

We compute the product LU :

$$(1.25 \ mark) LU = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21}u_{11} & \ell_{21}u_{12} + u_{22} & \ell_{21}u_{13} + u_{23} \\ \ell_{31}u_{11} & \ell_{31}u_{12} + \ell_{32}u_{22} & \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} \end{pmatrix}.$$

By identification with A, we obtain :

$$u_{11} = 7, \quad u_{12} = -3, \quad u_{13} = 1$$

$$\ell_{21} = \frac{-14}{7} = -2, \quad \ell_{31} = \frac{7}{7} = 1$$

$$u_{22} = 5 - \ell_{21}u_{12} = 5 - (-2)(-3) = -1$$

$$u_{23} = -3 - \ell_{21}u_{13} = -3 - (-2)(1) = -1$$

$$\ell_{32} = \frac{-4 - \ell_{31}u_{12}}{u_{22}} = \frac{-4 - (-3)}{-1} = 1$$

$$u_{33} = 4 - \ell_{31}u_{13} - \ell_{32}u_{23} = 4 - 1 - (-1) = 4$$

Thus :

$$(0.75\ mark)L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad (1.5\ mark)U = \begin{pmatrix} 7 & -3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 4 \end{pmatrix}.$$

3. Solution using LU decomposition

$$Ax = b \Leftrightarrow L\,(Ux) = b \Leftrightarrow \begin{cases} Ly = b & (1) \\ Lx = y & (2) \end{cases} \ (0.5\ mark)$$

$$Ly = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -8 \\ 13 \\ -3 \end{pmatrix} \Leftrightarrow \begin{cases} y_1 = -8 \\ -2y_1 + y_2 = 13 \\ y_1 + y_2 + y_3 = -3 \end{cases} \Leftrightarrow \begin{cases} y_1 = -8 \\ y_2 = -3 (1\ mark) \\ y_3 = 8 \end{cases}$$

$$Ux = y \Leftrightarrow \begin{pmatrix} 7 & -3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ 8 \end{pmatrix} \Leftrightarrow \begin{cases} 7x_1 - 3x_2 + x_3 = -8 \\ -x_2 - x_3 = -3 \\ 4x_3 = 8 \end{cases} \Leftrightarrow \begin{cases} x_1 = -1 \\ x_2 = 1 (1\ mark) \\ x_3 = 2 \end{cases}$$

Thus :

$$X = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

4. Determinant of A

Since $A = LU$, we have :

$$\det(A) = \det(L)\det(U).$$

Because L is unit lower triangular :

$$\det(L) = 1.$$

Thus :

$$\det(A) = 7 \cdot (-1) \cdot 4 = -28. \quad (1\ mark)$$

Exercise 3 (4 marks)

Let

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 7 \end{pmatrix}.$$

1. Write $A = D - E - F$, where D is diagonal, $-E$ strictly lower triangular, and $-F$ strictly upper triangular.
2. Determine the Jacobi iteration matrix $J = D^{-1}(E + F)$.
3. Compute
$$\|J\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |j_{ij}|$$
.
4. Using the obtained result, conclude whether the Gauss–Seidel method converges.

Solution :

We consider the matrix
$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 7 \end{pmatrix}.$$

1. Decomposition $A = D - E - F$
The diagonal matrix is :
$$(0.25mark) D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

The strictly lower triangular part is :
$$(0.25mark) - E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad .$$

The strictly upper triangular part is :
$$(0.25mark) - F = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus :
$$A = D - E - F.$$

2. Jacobi iteration matrix
We compute :
$$(0.5mark) E + F = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Since
$$(0.5mark) D^{-1} = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix},$$
$$(1.25mark) J = D^{-1}(E + F) = \begin{pmatrix} 0 & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{6} & 0 & -\frac{1}{6} \\ -\frac{1}{7} & -\frac{1}{7} & 0 \end{pmatrix}.$$

3. Computation of $\|J\|_\infty$
We compute the row sums :

$$\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

$$\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

$$\frac{1}{7}+\frac{1}{7}=\frac{2}{7}$$

Thus :

$$(0.5 mark)\|J\|_\infty=\max\left(\frac{2}{5},\frac{1}{3},\frac{2}{7}\right)=\frac{2}{5}<1.$$

4. Conclusion
Since :

$$\|J\|_\infty<1,$$

Hence, the Jacobi iterative method is convergent.(0.5 mark)

Module manager. Merini.A
Good luck