Mohamed Boudiaf University - M'sila **University Year : 2024/2025**
Faculty of Technology
Department of Civil Engineerin
Module : Numerical Methods

Sample solution the final exam

Exercice n°1 : (7 marks)

Consider the function $f(x)$ defined by :$f(x) = exp\,(x) - x - 5$

1. Show that $f(x) = 0$ has a unique solution α in the interval $[1; 2]$.
2. Verify that the function $g(x) = ln(x + 5)$ satisfies the conditions of the fixed point theorem in the interval $[1; 2]$.
3. Calculate the root α with $x_0 = 2$ and the precision $\epsilon = 5 \times 10^{-3}$ (take 4 digits after the decimal point).

Solution :

1. The function f is continuous on $\Re$ and so on the interval $[1; 2]$ (0.25 mark)
   $f(1) \times f(2) < 0$ (0.25 mark)
   $f'(x) = exp(x) - 1 > 0$, $\forall x \in [1; 2]$ (0.25 mark)
   So, according to the intermediate value theorem, there exists a unique $\alpha \in [1; 2]$. (0.25 mark)
2. **i)** $f(x) = 0 \Leftrightarrow exp\,(x) - x - 5 = 0 \Leftrightarrow exp\,(x) = x + 5 \Leftrightarrow x = ln(x + 5)$ $\Leftrightarrow g(x) = ln(x + 5)$ (1 mark)

   ii) Let $x \in [1; 2]$, i.e., $1 \leq x \leq 2 \Leftrightarrow 6 \leq x + 5 \leq 7 \Leftrightarrow ln6 \leq ln(x + 5) \leq ln7 \Leftrightarrow 1.7918 \leq g(x) \leq 1.9459$, so, $\forall x \in [1; 2] : g(x) \in [1; 2]$. (1 mark)

   iii) $|g'(x)| = |\dfrac{1}{x+5}| \leq \dfrac{1}{6} < 1, \forall x \in [1; 2]$ (0.5 mark)
   So, the function $g(x)$ satisfies the conditions of the fixed point theorem in the interval $[1; 2]$, and the sequence

$$\begin{cases} x_0 \in [1; 2] \quad (0.5\ mark) \\ x_{n+1} = g\,(x_n) \end{cases}$$

   converges to the solution α.
3. For $x_0 = 2$ and $\epsilon = 5 \times 10^{-3}$, we obtain :

$$\begin{aligned} x_1 &= g\,(x_0) = \ln(2 + 5) = 1.9459, \quad |x_1 - x_0| = 0.0541 > \epsilon \quad (0.75\ mark) \\ x_2 &= g\,(x_1) = \ln(x_1 + 5) = 1.9382, \quad |x_2 - x_1| = 0.0077 > \epsilon \quad (0.75\ mark) \\ x_3 &= g\,(x_2) = \ln(x_2 + 5) = 1.9370; \quad |x_3 - x_2| = 0.0012 < \epsilon \quad (0.75\ mark) \end{aligned}$$

   So we take $\alpha = 1.937$ (0.75 mark)

Exercice n°2 (9 marks)

Consider the following linear system (S) :

$$(S) \begin{cases} x_1 - 3x_2 = 28 \\ -3x_1 + 13x_2 - 8x_3 = 96 \\ -8x_2 + 41x_3 = -985 \end{cases}$$

i) Write the system (S) as $AX = b$.

ii) Verify that the matrix A is symmetric positive definite.

iii) Calculate the Cholesky factorisation of $A = MM^t$.

iv) Solve system (S) using Cholesky's factorisation.

v) Deduce the determinant of A.

Solution :

i)

$$(S) \Longleftrightarrow Ax = b \Longleftrightarrow \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 96 \\ -985 \end{pmatrix} \text{ (1 } mark\text{)}$$

ii) The matrix A is symmetric :

$$A^t = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} = A \quad (0.25\ mark)$$

The matrix A is positive definite :

$\det A_{[1]} = 1 > 0$, $\det A_{[2]} = \det \begin{pmatrix} 1 & -3 \\ -3 & 13 \end{pmatrix} = 4 > 0$ and $\det A_{[3]} = \det \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} = 100 > 0$. (0.75 mark)

iii) The matrix A is a positive definite symmetric matrix, it has decomposition $A = MM^t$.(0.5 mark)

We put

$$M = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \text{ such that } m_{ii} > 0, \forall i \in \{1, 2, 3\}$$

$$\begin{aligned} A &= MM^t \Leftrightarrow \\ \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} &= \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \times \begin{pmatrix} m_{11} & m_{21} & m_{31} \\ 0 & m_{22} & m_{32} \\ 0 & 0 & m_{33} \end{pmatrix} \\ &= \begin{pmatrix} m_{11}^2 & m_{11}m_{21} & m_{11}m_{31} \\ m_{11}m_{21} & m_{21}^2 + m_{22}^2 & m_{21}m_{31} + m_{22}m_{32} \\ m_{11}m_{31} & m_{21}m_{31} + m_{22}m_{32} & m_{31}^2 + m_{32}^2 + m_{33}^2 \end{pmatrix} \text{ (1 } mark\text{)} \end{aligned}$$

Hence

$$\begin{cases} m_{11} = \sqrt{1} = 1 \\ m_{11}m_{21} = -3 \Rightarrow m_{21} = -3 \\ m_{11}m_{31} = 0 \Rightarrow m_{31} = 0 \end{cases} \text{ et } \begin{cases} m_{21}^2 + m_{22}^2 = 13 \Rightarrow m_{22} = \sqrt{13 - 9} = 2 \\ m_{21}m_{31} + m_{22}m_{32} = -8 \Rightarrow m_{32} = -4 \\ m_{31}^2 + m_{32}^2 + m_{33}^2 = 41 \Rightarrow m_{33} = 5 \end{cases}$$

Finally :

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & -4 & 5 \end{pmatrix} \text{ (1.5 } mark\text{) and } M^t = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 5 \end{pmatrix} \text{ (0.5 } mark\text{)}$$

iv) The resolution of the system $AX = b$ becomes

$$Ax = b \Leftrightarrow M\left(M^t x\right) = b \Leftrightarrow \left\{ \begin{array}{ll} My = b & (1) \\ M^t x = y & (2) \end{array} \right. \quad (0.5\ mark)$$

$$My = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & -4 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 96 \\ -985 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} y_1 = 28 \\ -3y_1 + 2y_2 = 96 \\ -4y_2 + 5y_3 = -985 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y_1 = 28 \\ y_2 = 90 \\ y_3 = -125 \end{array} \right.$$

(1 mark)

$$M^t x = y \Leftrightarrow \begin{pmatrix} 1 & -3 & 5 \\ 0 & 2 & -4 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 90 \\ -125 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{c} x_1 - 3x_2 = 28 \\ 2x_2 - 4x_3 = 90 \\ 5x_3 = -125 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x_1 = 13 \\ x_2 = -5 (1\ mark) \\ x_3 = -25 \end{array} \right.$$

v)

$$\det(A) = \det MM^t = \det(M) \times \det(M^t) = [\det(M)]^2 = \prod_{i=1}^{n} m_{ii}^2 = 1^2 \times 2^2 \times 5^2 = 100. (1\ mark)$$

<u>Exercice n°</u>3 (4 marks)

Consider the points given in the table below :

x_i	0	2	3	5
$f(x_i)$	-7	-13	5	143

a) Construct the divided differences table.

b) Find the equation of Newton's polynomial passing the first three points.

c) Deduce the equation of Newton's polynomial of degree three passing through all the points.

<u>Solution :</u>

a) Divided differences table : (1.5 mark)

x_i	$f(x_i)$	$DD1$	$DD2$	$DD3$
$x_0 = 0$	$f(x_0) = -7 = a_0$			
$x_1 = 2$	$f(x_1) = -13$	$\frac{-13+7}{2-0} = -3 = a_1$		
$x_2 = 3$	$f(x_2) = 5$	$\frac{5-(-13)}{3-2} = 18$	$\frac{18-(-3)}{3-0} = 7 = a_2$	
$x_3 = 5$	$f(x_3) = 143$	$\frac{143-5}{5-3} = 69$	$\frac{69-18}{5-2} = 17$	$\frac{17-7}{5-0} = 2 = a_3$

b)

$$\begin{aligned} P_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= -7 - 3(x - 0) + 7(x - 0)(x - 2) \\ &= 7x^2 - 17x - 7 \quad (1\ mark) \end{aligned}$$

c)

$$\begin{aligned} P_3(x) &= P_2(x) + a_3 (x - x_0)(x - x_1)(x - x_2) \quad (0.5\ mark) \\ &= 7x^2 - 17x - 7 + 2(x-0)(x-2)(x-3) \\ &= 2x^3 - 3x^2 - 5x - 7 \quad (1\ mark) \end{aligned}$$

Module manager. Merini.A

Good luck