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Numerical Methods Course
Chapter 1: Solving non-linear equations $f(x) = 0$

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1 Introduction

Numerical Methods represent the essential link between abstract mathematical theories and real-world engineering and physical applications. In many cases, the mathematical modeling of complex problemssuch as soil mechanics equations or nonlinear electrical circuit responsesfaces the impossibility of finding exact **Analytical Solutions**. This is where Numerical Analysis intervenes, providing approximate yet highly accurate solutions suitable for computer processing.

Why study Numerical Methods? The importance of this field lies in providing engineers with tools to:

Solve **Nonlinear Equations** that are algebraically impossible to crack.

Handle **Experimental Data** through interpolation and curve fitting.

- Perform **Numerical Integration and Differentiation** for complex functions.
- Solve large systems of **Linear Equations**, which form the backbone of engineering simulation software.

The course consists of the following key topics:

1) Introduction to Calculation Errors and Approximations.

2) Introduction to Methods for Solving Nonlinear Equations.

3) Bisection Method.

4) Fixed-Point Iteration Method (Successive Approximations).

5) Newton-Raphson Method

2 Introduction to calculation errors and approximations

An important part of numerical analysis is the study and evaluation of errors in order to reduce them. These errors can be classified into three categories:
1) errors on the data.
2) rounding errors.
3) approximation errors.

2.1 Absolute and relative errors

Let x be an exact value and x^* be its approximate value. We write:

$$x \approx x^* \quad \text{or} \quad x \simeq x^* \tag{1}$$

2.1.1 Absolute error

Definition 2.1 Absolute error

The absolute error, denoted by $\Delta(x)$, is given by:

$$\Delta(x) = |x - x^*| \tag{2}$$

Corollary

i) The smaller the absolute error, the more precise the approximation x^*.

ii) $\Delta(x) \geq 0$.

Example 2.1

Let the exact value be $x = 17$, and the measured values be $x_1^* = 16.32$ and $x_2^* = 17.02$. Then we have:

$$\Delta_1(x) = |x - x_1^*| = |17 - 16.32| = 0.68$$

$$\Delta_2(x) = |x - x_2^*| = |17 - 17.02| = 0.02$$

Since $\Delta_2(x) < \Delta_1(x)$, then $x_2^* = 17.02$ is the most precise value (the nearest to x).

2.1.2 Relative error

Definition 2.2 Relative error

The relative error, denoted by r_x, is given by:

$$r_x = \frac{\Delta(x)}{|x|} \tag{3}$$

The relative error is often expressed as a percentage, i.e., $r_x \times 100\%$.

Example 2.2

Using the previous example where $x^* = 17.02$ is an approximate value of $x = 17$, we have:

$$r_x = \frac{\Delta(x)}{|x|} = \frac{0.02}{17} \approx 1.176 \times 10^{-3} \approx 0.12\%$$

2.1.3 Absolute and Relative Error Bounds (Majorants)

In practice, it is difficult to evaluate the absolute error because x is often unknown. Therefore, we introduce the concept of the **upper bound** (or limit) of the error, denoted by Δx.

Remark

The absolute error satisfies: if x is unknown

$$\Delta(x) = |x - x^*| \leq \Delta x \tag{4}$$

Thus, we can write:

$$x = x^* \pm \Delta x \tag{5}$$

which means that $x \in [x^* - \Delta x, x^* + \Delta x]$.

Remark

1) Since $x \approx x^*$, in practice we take $r_x = \frac{\Delta x}{|x^*|}$, which is an **upper bound of the relative error** of x^*, and we write $x = x^* \pm |x^*| r_x$.
2) In the absence of the exact error, Δx (or r_x) is commonly referred to as the absolute (or relative) error of x^*.

2.2 Decimal representation of an approximate number :

Definition 2.3

Any positive number x^* can be expressed as :

$$x^* = \alpha_m.10^m + \alpha_{m-1}.10^{m-1} + ... + \alpha_{m-n+1}.10^{m-n+1} + ... \qquad (6)$$

with $\alpha_m \neq 0$, $\alpha_i \in \{0, 1, 2, ..., 9\}$,where m is the highest rank of x^* (the most greatest power of 10)

Example 2.3

$29,403 = 2 \times 10^1 + 9 \times 10^0 + 4 \times 10^{-1} + 0 \times 10^{-2} + 3 \times 10^{-3}$, we have $m = 1$,
$0,00120 = 1 \times 10^{-3} + 2 \times 10^{-4} + 0 \times 10^{-5}$, on a $m = -3$.
$\pi = 3.14159265358... = 3.10^0 + 1.10^{-1} + 4.10^{-2} + 1.10^{-3} + ...$

2.3 Significant Digits

Definition 2.4 Significant Digits

The digits $\alpha_m, \alpha_{m-1}, \ldots, \alpha_{m-n+1}$ are called **significant digits** (or **significant figures**). In other words, a significant digit of a number is any digit in its decimal representation, starting from the first non-zero digit on the left and ending with the last digit on the right (including zeros).

Example 2.4

29.403 has 5 **significant digits**.

0.00120 has 3 **significant digits**.

Remark

The number of significant digits in $x = 25100$ is **ambiguous** unless written in scientific notation, as it could represent:

- $x = 2.51 \times 10^4$ (3 significant digits).
- $x = 2.510 \times 10^4$ (4 significant digits).
- $x = 2.5100 \times 10^4$ (5 significant digits).

2.4 Correct Significant Digits (csd)

Definition 2.5 csd

We say that the first n significant digits of an approximate number x^* are **correct** (or **valid**) if the absolute error Δx satisfies the following condition:

$$\Delta x \leq 0.5 \times 10^{m-n+1}$$

where m is the exponent of the first significant digit (the power of 10 associated with the first non-zero digit α_m).

Properties

a) If a certain significant digit is correct, all digits to its left are also correct.

b) If a certain digit is not correct, all digits to its right are also not correct.

Example 2.5

Let $x = 223.864$ and its approximation $x^* = 223.887$. Since the first digit is in the hundreds place (10^2), we have $m = 2$. The absolute error is calculated as:

$$\Delta x = |223.864 - 223.887| = 0.023 = 0.23 \times 10^{-1}$$

Testing the condition $\Delta x \leq 0.5 \times 10^{m-n+1}$, we find:

- $0.23 \times 10^{-1} \leq 0.5 \times 10^{-1} \implies m - n + 1 = -1$
- Substituting $m = 2$: $2 - n + 1 = -1 \implies 3 - n = -1 \implies n = 4$

Thus, the first four significant digits $(2, 2, 3,$ and $8)$ are **correct**.

2.5 Relation between Relative Error and CSD

Proposition 2.1

If an approximate number x^* has n correct significant digits, then its relative error r_x satisfies the following inequality:

$$r_x < \frac{1}{\alpha_m} \times 10^{-(n-1)}$$

where α_m is the first significant digit. In a more general (and simpler) form, this is often expressed as:

$$r_x < 5 \times 10^{-n}$$

Proof

Proof. By definition, if an approximate number x^* has n correct significant digits, the absolute error Δx satisfies:

$$\Delta x \leq 0.5 \times 10^{m-n+1} \tag{7}$$

The approximate number x^* can be written in scientific notation as:

$$|x^*| \geq \alpha_m \times 10^m \tag{8}$$

where α_m is the first significant digit ($1 \leq \alpha_m \leq 9$). The relative error r_x is defined as $r_x = \frac{\Delta x}{|x^*|}$. Substituting the inequalities (1) and (2), we get:

$$r_x \leq \frac{0.5 \times 10^{m-n+1}}{\alpha_m \times 10^m} = \frac{0.5}{\alpha_m} \times 10^{-n+1} = \frac{5}{\alpha_m} \times 10^{-n}$$

Since the minimum value of α_m is 1, we have:

$$r_x < 5 \times 10^{-n}$$

Alternatively, we can express it as:

$$r_x < \frac{1}{\alpha_m} \times 10^{-(n-1)}$$

This completes the proof.

Remark

This relation shows that the more correct significant digits a number has, the smaller its relative error. For instance, if $n = 4$, the relative error is guaranteed to be less than 0.0005 (or 0.05%).

2.6 Rules of Rounding-Off

Let $x = \beta_1\beta_2 \dots \beta_n\beta_{n+1} \dots$, where $\beta_i \in \{0, 1, \dots, 9\}$ and $\beta_1 \neq 0$. The rounding of x to n significant digits is performed according to the following rules:

1) If $\beta_{n+1} > 5$, then the n-th digit is incremented: $\beta_n^* = \beta_n + 1$.
2) If $\beta_{n+1} < 5$, then the n-th digit remains unchanged: $\beta_n^* = \beta_n$.
3) If $\beta_{n+1} = 5$, there are two cases:
   - If there is at least one non-zero digit among $\beta_{n+2}, \beta_{n+3}, \dots,$ then $\beta_n^* = \beta_n + 1$.
   - If all digits following β_{n+1} are zeros ($\beta_{n+2} = \beta_{n+3} = \cdots = 0$):
     - If β_n is **even**, it remains unchanged: $\beta_n^* = \beta_n$.
     - If β_n is **odd**, it is incremented: $\beta_n^* = \beta_n + 1$.

Example 2.6

1) Round $x = 0.254$ to two significant digits: $\beta_3 = 4 < 5$, so $x^* = 0.25$.
2) Round $x = 0.4368$ to three significant digits: $\beta_4 = 8 > 5$, so $x^* = 0.437$.
3) Round $x = 1.534500$ to four significant digits: $\beta_5 = 5$ followed by zeros. Since $\beta_4 = 4$ is even, $x^* = 1.534$.
4) Round $x = 1.5347500$ to five significant digits: $\beta_6 = 5$ followed by zeros. Since $\beta_5 = 7$ is odd, $x^* = 1.5348$.
5) Round $x = 23.6050420$ to four significant digits: $\beta_5 = 5$ is followed by non-zero digits $(4, 2)$, so $x^* = 23.61$.

Remark

The rounding error, denoted by Δ_{arr}, satisfies the estimate:

$$\Delta_{arr} \leq 0.5 \times 10^{m-n+1} \tag{9}$$

After rounding, the total uncertainty of the approximate value is:

$$x = x^*_{arr} \pm (\Delta_{arr} + \Delta x) \tag{10}$$

In general, if we assume $\Delta_{arr} \approx \Delta x$, then:

$$x = x^*_{arr} \pm 2\Delta x$$

Remark

1) Only the final result should be rounded; never round intermediate results.
2) A final result is only as accurate as the least accurate data used.
3) A correctly rounded number consists only of **correct significant digits**.

2.7 Error Propagation

2.7.1 A General Error Formula

Definition 2.6

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function of n variables $(x_1, \ldots, x_n)$. The absolute error Δf and relative error r_f are given by:

$$\Delta f \approx \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \Delta x_i = \left| \frac{\partial f}{\partial x_1} \right| \Delta x_1 + \cdots + \left| \frac{\partial f}{\partial x_n} \right| \Delta x_n \tag{11}$$

$$r_f = \frac{\Delta f}{|f^*|} \tag{12}$$

Example 2.7 C

nsider the function $f(x, y) = x^2 \sin(y)$. Suppose we have the following measurements:

- $x^* = 3.0$ with an absolute error $\Delta x = 0.1$.
- $y^* = \frac{\pi}{6}$ (approx 0.5236) with an absolute error $\Delta y = 0.02$.

Calculate the absolute error of the function Δf at this point.

Solution

According to the **General Error Formula**, the absolute error Δf is:

$$\Delta f \approx \left|\frac{\partial f}{\partial x}\right| \Delta x + \left|\frac{\partial f}{\partial y}\right| \Delta y$$

Step 1: Calculate the partial derivatives:

1) $\frac{\partial f}{\partial x} = 2x\sin(y)$
2) $\frac{\partial f}{\partial y} = x^2\cos(y)$

Step 2: Evaluate the derivatives at (x^*, y^*):

- $\frac{\partial f}{\partial x}\Big|_{(3,\pi/6)} = 2(3)\sin(\frac{\pi}{6}) = 6 \times 0.5 = 3.0$
- $\frac{\partial f}{\partial y}\Big|_{(3,\pi/6)} = (3)^2\cos(\frac{\pi}{6}) = 9 \times \frac{\sqrt{3}}{2} \approx 9 \times 0.866 = 7.794$

Step 3: Apply the error formula:

$$\Delta f \approx |3.0| \times (0.1) + |7.794| \times (0.02)$$

$$\Delta f \approx 0.3 + 0.15588 = 0.45588$$

Result: The absolute error of the function is approximately $\Delta f \approx 0.456$.

Remark

This example shows that the error in y is magnified by a factor of 7.794, while the error in x is magnified by a factor of 3.0. This indicates that the function is more sensitive to errors in y at this specific point.

2.7.2 Addition and Subtraction Errors

Proposition 2.2 T

e propagation of errors in addition and subtraction follows these rules:

- **Absolute Error:** $\Delta_{x\pm y} = \Delta x + \Delta y$
- **Relative Error (Addition):** $r_{x+y} \leq \max(r_x, r_y)$ (for $x, y > 0$).
- **Relative Error (Subtraction):** $r_{x-y} = \frac{\Delta x + \Delta y}{|x-y|}$.

Proof

Proof. We use the **General Error Formula** for a function $f(x, y)$:

$$\Delta f \approx \left|\frac{\partial f}{\partial x}\right| \Delta x + \left|\frac{\partial f}{\partial y}\right| \Delta y$$

1. Absolute Error for Addition and Subtraction: Let $f(x, y) = x \pm y$. The partial derivatives are:

- $\frac{\partial f}{\partial x} = 1$
- $\frac{\partial f}{\partial y} = \pm 1$

Applying the formula:

$$\Delta(x \pm y) = |1|\Delta x + |\pm 1|\Delta y = \Delta x + \Delta y$$

This shows that the absolute error of a sum or difference is the **sum of the absolute errors**.
2. Relative Error for Subtraction: By definition, the relative error is $r_f = \frac{\Delta f}{|f|}$. For $f = x - y$:

$$r_{x-y} = \frac{\Delta x + \Delta y}{|x - y|}$$

Note: If $x \approx y$, the denominator $|x - y|$ becomes very small, causing r_{x-y} to become very large (Loss of Significance).
3. Relative Error for Addition (Upper Bound): For $f = x + y$ (where $x, y > 0$):

$$r_{x+y} = \frac{\Delta x + \Delta y}{x + y} = \frac{x \cdot r_x + y \cdot r_y}{x + y}$$

This is a weighted average of r_x and r_y. Thus:

$$r_{x+y} \leq \max(r_x, r_y)$$

Example 2.8

Let $x^* = 255$ and $y^* = 250$ be approximate numbers with relative errors $r_x = r_y = 0.1\% = 10^{-3}$.
1. Calculate Absolute Errors:

- $\Delta x = x^* \cdot r_x = 255 \times 10^{-3} = 0.255$
- $\Delta y = y^* \cdot r_y = 250 \times 10^{-3} = 0.250$

2. Error in Addition ($x + y = 505$):

- **Absolute Error:** $\Delta_{x+y} = \Delta x + \Delta y = 0.255 + 0.250 = 0.505$
- **Relative Error:** $r_{x+y} = \frac{0.505}{505} = 0.001 = 0.1\%$

3. Error in Subtraction ($x - y = 5$):

- **Absolute Error:** $\Delta_{x-y} = \Delta x + \Delta y = 0.505$
- **Relative Error:** $r_{x-y} = \frac{0.505}{5} = 0.101 = 10.1\%$

2.7.3 Multiplication and Division Errors

Proposition 2.3

For both multiplication ($f = xy$) and division ($f = x/y$), the relative error of the result is the sum of the relative errors of the individual components:

$$r_f = r_x + r_y$$

Proof

Proof. **1. For Multiplication ($f = xy$):** Using the GEF:

$$\Delta f = \left|\frac{\partial f}{\partial x}\right| \Delta x + \left|\frac{\partial f}{\partial y}\right| \Delta y = |y|\Delta x + |x|\Delta y$$

The relative error is then:

$$r_f = \frac{\Delta f}{|xy|} = \frac{|y|\Delta x}{|xy|} + \frac{|x|\Delta y}{|xy|} = \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} = r_x + r_y$$

Proof

Proof. **2. For Division ($f = x/y$):** The partial derivatives are $\frac{\partial f}{\partial x} = \frac{1}{y}$ and $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$.

$$\Delta f = \left|\frac{1}{y}\right| \Delta x + \left|-\frac{x}{y^2}\right| \Delta y = \frac{\Delta x}{|y|} + \frac{|x|\Delta y}{y^2}$$

The relative error is:

$$r_f = \frac{\Delta f}{|x/y|} = \left(\frac{\Delta x}{|y|} + \frac{|x|\Delta y}{y^2}\right) \cdot \frac{|y|}{|x|} = \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} = r_x + r_y$$

2.7.4 Power Errors

Proposition 2.4

For a function of the form $f(x) = x^n$, where n is a constant, the propagation of errors is described as follows:

- **Absolute Error:** $\Delta(x^n) = |n| \cdot |x^{n-1}|\Delta x$
- **Relative Error:** $r_{x^n} = |n| \cdot r_x$

Proof

Proof. Using the general error formula for a single variable function $f(x)$:

$$\Delta f = \left|\frac{df}{dx}\right| \Delta x$$

For $f(x) = x^n$, the derivative is $\frac{df}{dx} = nx^{n-1}$. Substituting this into the absolute error formula:

$$\Delta(x^n) = |nx^{n-1}|\Delta x = |n| \cdot |x^{n-1}|\Delta x$$

To find the relative error r_{x^n}, we divide by $|f(x)| = |x^n|$:

$$r_{x^n} = \frac{\Delta(x^n)}{|x^n|} = \frac{|n| \cdot |x^{n-1}|\Delta x}{|x^n|} = |n| \cdot \frac{\Delta x}{|x|} = |n| \cdot r_x$$

Remark

This result is very significant: it shows that the relative error is multiplied by the **exponent**. For example, if x has a 1% error, then x^3 will have a 3% error, and $\sqrt{x}$ (where $n = 1/2$) will have only a 0.5% error.

Corollary

For a function involving both multiplication and division, such as $f = \frac{x^\alpha y^\beta}{z^\gamma}$, the relative error is simply the sum of the individual relative errors:

$$r_f = |\alpha|r_x + |\beta|r_y + |\gamma|r_z$$

2.7.5 Practical Example

Example 2.9

Let $x = 0.1256$ be given with an error of 0.5%..
1) Give the absolute error of x.
2) Determine the exact number of significant digits of this number.
3) Round the result to the last csd .
Solution

1) **Absolute Error:**

$$\Delta x = |x| \cdot r_x = 0.1256 \times 0.005 = 6.28 \times 10^{-4}$$

2) **Correct Significant Digits (n):** The first significant digit is 1, in the 10^{-1} position ($m = -1$). Condition: $\Delta x \leq 0.5 \times 10^{m-n+1}$

$$6.28 \times 10^{-4} \leq 0.5 \times 10^{-n}$$

So, $0.0628 \times 10^{-2} \leq 0.5 \times 10^{-n}$.

Hence, $n = 2$.

3) **Rounding:** Rounding $x = 0.1256$ to $n = 2$ digits gives $x^*_{arr} = 0.13$. Final result: $x = 0.13 \pm 0.01$.

3 Methods for Solving Non-linear Equations

In this section, we focus on numerical techniques for finding the roots of a function of a single variable.

Definition 3.1 Root of an Equation

Let $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that $\alpha \in D(f)$ is a **root** of the equation $f(x) = 0$ if:

$$f(\alpha) = 0$$

3.1 Root Separation (Isolation)

Before applying numerical methods, we must isolate the root.

Definition 3.2 Isolated Root

A root α of the equation $f(x) = 0$ is said to be **isolated** (or separated) in an interval $[a, b]$ if α is the unique root of this equation within that interval. That is:

$$\forall x \in [a, b], \quad f(x) = 0 \iff x = \alpha$$

In practice, two main approaches are used to isolate roots:

3.1.1 Graphical Method

This method involves visualizing the function to estimate the location of its roots.

Two techniques are commonly applied:

a) **Direct Plotting:** Draw $y = f(x)$ and find its intersections with the x-axis (Ox).

b) **Decomposition:** Decompose f into $f = f_1 - f_2$ and find the intersection points of $y = f_1(x)$ and $y = f_2(x)$.

Remark

Standard curves (lines, parabolas) are preferred for f_1 and f_2.

Example 3.1 Graphical Root Isolation

Consider the function $f(x) = x^3 + x - 1$ in $[0, 1]$. By setting $f_1(x) = x^3$ and $f_2(x) = 1 - x$, we find their intersection.

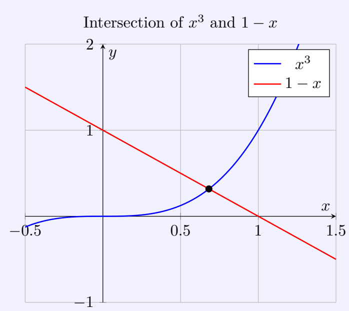


Figure 1: Root α located in $[0, 1]$.

3.1.2 Scanning Method (Incremental Search)

This method consists of evaluating the sign of the function $f(x)$ at different points within a given range to detect sign changes.

A

ccording to the **Intermediate Value Theorem**, if $f(x)$ is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists at least one root $\alpha \in]a, b[$. The scanning method systematically checks sub-intervals to find these sign changes.

Example 3.2 Root Separation by Scanning

Consider the polynomial $P(x) = x^4 - x^3 - x^2 + 7x - 6$. We can analyze the sign of $P(x)$ at several points:

x	-4	-3	-1	0	2
Sign of $P(x)$	$+$	$+$	$-$	$-$	$+$

By observing the sign changes in the table, we conclude that $P(x)$ has at least two real roots:

- $\alpha_1 \in]-3, -1[$ because $P(-3) > 0$ and $P(-1) < 0$.
- $\alpha_2 \in]0, 2[$ because $P(0) < 0$ and $P(2) > 0$.

4 Bisection Method (Dichotomy)

The bisection principle is based on the **Intermediate Value Theorem**:

Definition 4.1 Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function on a closed interval $[a, b]$. If $f(a) \times f(b) < 0$, then there exists at least one root $\alpha \in]a, b[$ such that $f(\alpha) = 0$.

Remark

- The condition $f(a) \times f(b) < 0$ implies that $f(a)$ and $f(b)$ have opposite signs.
- The continuity of f is essential for this method to work.
- If f is also **strictly monotone** on $[a, b]$, then the root α is unique.

4.1 The Algorithm

Bisection Procedure

To find an approximation of a root α in the interval $[a_0, b_0]$:

1) Calculate the midpoint: $x_k = \frac{a_k+b_k}{2}$.
2) If $f(x_k) = 0$, then $\alpha = x_k$ (Exact solution found).
3) If $f(x_k) \neq 0$:
   - If $f(a_k) \times f(x_k) < 0$, the root lies in the left half: $[a_{k+1}, b_{k+1}] = [a_k, x_k]$.
   - If $f(a_k) \times f(x_k) > 0$, the root lies in the right half: $[a_{k+1}, b_{k+1}] = [x_k, b_k]$.
4) Repeat until the stopping criterion is met.

4.2 Geometric Interpretation

The bisection method is a **bracketed method** that relies on the geometry of the function's curve.

Geometric Concept

Geometrically, the method starts with two points a_0 and b_0 where the curve $y = f(x)$ lies on opposite sides of the x-axis. By calculating the midpoint x_0, we effectively **halve the search area**.

- If the curve crosses the axis between a_0 and x_0, we discard the right half.
- Otherwise, we discard the left half.

This process creates a sequence of nested intervals that "squeeze" the root α from both sides.

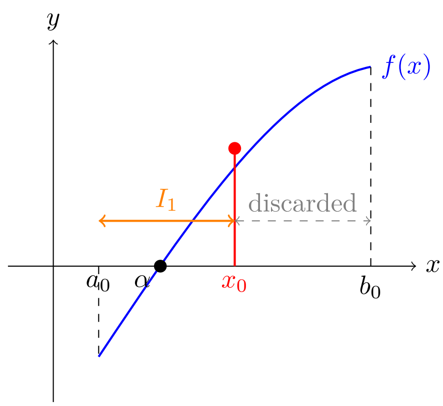


Figure 2: Geometric visualization of the interval halving process.

4.3 Error Estimation and Stopping Criteria

The absolute error at the k-th iteration is bounded by:

$$|\alpha - x_k| \leq \frac{b_0 - a_0}{2^{k+1}} \tag{13}$$

Proposition 4.1 Number of Iterations

To achieve an accuracy ϵ, the required number of iterations n must satisfy:

$$n \geq \frac{\log\left(\frac{b_0 - a_0}{2\epsilon}\right)}{\log 2}$$

4.4 Application Exercise: Bisection Method

Example 4.1 Solving $x^3 - x - 1 = 0$ with High Precision

Let $f(x) = x^3 - x - 1$. Use the Bisection method to find the solution in the interval $[1, 2]$ with an accuracy of $\epsilon = 10^{-3}$.

Solution

1. Theoretical Number of Iterations: To ensure an error less than $\epsilon = 0.001$, the number of iterations n must satisfy:

$$n \geq \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln(2)} = \frac{\ln(1000)}{\ln(2)} \approx 9.96$$

Therefore, we need at least $n = 10$ iterations.
2. Iteration Table: We summarize the first few and the final steps of the process:

k	a_k	b_k	x_k (Midpoint)	$f(x_k)$	Error $\frac{b_k - a_k}{2}$
0	1.0000	2.0000	1.5000	0.8750 (+)	0.5000
1	1.0000	1.5000	1.2500	-0.2969 (-)	0.2500
2	1.2500	1.5000	1.3750	0.2246 (+)	0.1250
3	1.2500	1.3750	1.3125	-0.0515 (-)	0.0625
4	1.3125	1.3750	1.3438	0.0826 (+)	0.0313
...	...	...	...	...	...
9	1.3242	1.3262	**1.3252**	0.0025 (+)	**0.0009**

3. Final Result: At the 10th iteration ($k = 9$), the error 0.0009 is less than $\epsilon = 0.001$. The approximate root of the equation is:

$$\alpha \approx \mathbf{1.325}$$

4.5 Advantages and Disadvantages of the Bisection Method

1) Advantages

- ✓ **Guaranteed Convergence .**
- ✓ **Error Control.**

2) Disadvantages

- × **Slow Convergence.**
- × **Sign Change Requirement.**

5 Successive Approximation Method (Fixed Point)

5.1 Principle

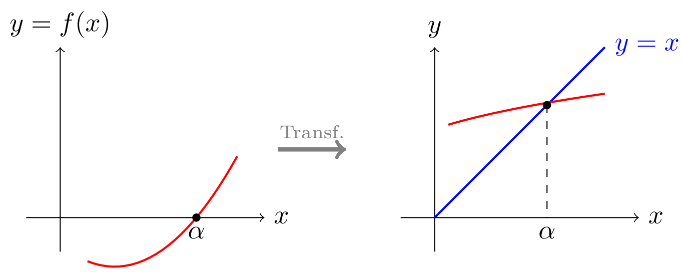
Definition 5.1 The Fixed-Point Principle

The fixed-point method consists of transforming the original equation $f(x) = 0$ into an equivalent form:

$$x = g(x) \tag{14}$$

A value α such that $g(\alpha) = \alpha$ is called a **fixed point**. The solution is found by constructing a sequence $(x_n)_{n\in\mathbb{N}}$ using the recurrence relation:

$$x_{n+1} = g(x_n), \quad \text{with } x_0 \text{ as an initial guess.} \tag{15}$$



(a) Finding root of $f(x) = 0$ (b) Finding fixed point $g(x) = x$

Example 5.1 Possible $g(x)$ Transformations

For the equation $x^2 - x - 2 = 0$, possible fixed-point forms include:

- $g_1(x) = x^2 - 2$ (Divergent near the root $\alpha = 2$)
- $g_2(x) = \sqrt{x+2}$ (Convergent)
- $g_3(x) = 1 + \frac{2}{x}$ (Convergent)

Key Question

What conditions must the function $g(x)$ satisfy for this fixed point α to exist and be unique within the interval $[a, b]$?

5.2 Fixed-Point Theorem and its Proof

Proposition 5.1 Fixed-Point Theorem

et g be a continuous function on the closed interval $[a, b]$ satisfying:

1) **Stability:** $g(x) \in [a, b]$ for all $x \in [a, b]$.
2) **Contraction:** g is differentiable on $]a, b[$ and there exists $M \in (0, 1)$ such that:
$$|g'(x)| \leq M < 1, \quad \forall x \in]a, b[$$

Then g has a **unique** fixed point $\alpha \in [a, b]$, and the sequence $x_{n+1} = g(x_n)$ converges to α for any $x_0 \in [a, b]$.

Proof

Proof. **1. Uniqueness:** Assume two distinct fixed points $\alpha, \beta \in [a, b]$. By the **Mean Value Theorem**: $|\alpha - \beta| = |g(\alpha) - g(\beta)| = |g'(\xi)| \cdot |\alpha - \beta|$. Since $|g'(\xi)| \leq M < 1$, then $|\alpha - \beta| \leq M|\alpha - \beta|$, which implies $(1 - M)|\alpha - \beta| \leq 0$. This is only possible if $\alpha = \beta$.
2. Convergence: Using MVT: $|x_{n+1} - x_n| \leq M|x_n - x_{n-1}| \leq M^n|x_1 - x_0|$. For $p > n$, the triangle inequality leads to: $|x_p - x_n| \leq \frac{M^n}{1-M}|x_1 - x_0|$. As $n \to \infty$, $M^n \to 0$, so (x_n) is a **Cauchy sequence** in $\mathbb{R}$, thus it converges to α.

5.3 Arrest Test (Stopping Criterion)

The minimum number of iterations n required to reach a precision ϵ is given by:

$$n \geq \left\lceil \frac{\ln\left(\frac{\epsilon(1-M)}{|x_1 - x_0|}\right)}{\ln M} \right\rceil$$

5.4 Numerical Application: Fixed-Point Method

Example 5.2 Solving $x^3 - x - 1 = 0$

Find the positive root of $f(x) = x^3 - x - 1 = 0$ in the interval $[1, 2]$ using the fixed-point method with a precision of $\epsilon = 0.01$.

Solution

1. Transformation and Verification of Conditions: We rewrite the equation as $x^3 = x + 1$, which gives the form $x = (x + 1)^{1/3}$. Let $g(x) = (x + 1)^{1/3}$.

- **Stability Condition:** For $x \in [1, 2]$, we have: $g(1) = \sqrt[3]{2} \approx 1.26 \in [1, 2]$ and $g(2) = \sqrt[3]{3} \approx 1.44 \in [1, 2]$. Since g is increasing, $g(x) \in [1, 2]$ for all $x \in [1, 2]$. (Condition satisfied).
- **Contraction Condition:** The derivative is $g'(x) = \frac{1}{3(x+1)^{2/3}}$. On $[1, 2]$, $|g'(x)|$ is a decreasing function, its maximum value is at $x = 1$:

$$M = \max_{x \in [1,2]} |g'(x)| = |g'(1)| = \frac{1}{3\sqrt[3]{4}} \approx 0.21 < 1$$

  Since $M < 1$, the uniqueness of the fixed point and the convergence of the sequence are guaranteed.

2. Iterative Calculation ($x_0 = 1.5$):

- **Iteration 1:** $x_1 = g(1.5) = (2.5)^{1/3} \approx 1.3572$
  Error: $|x_1 - x_0| = |1.3572 - 1.5| = 0.1428 > \epsilon$
- **Iteration 2:** $x_2 = g(1.3572) = (2.3572)^{1/3} \approx 1.3309$
  Error: $|x_2 - x_1| = |1.3309 - 1.3572| = 0.0263 > \epsilon$
- **Iteration 3:** $x_3 = g(1.3309) = (2.3309)^{1/3} \approx 1.3259$
  Error: $|x_3 - x_2| = |1.3259 - 1.3309| = 0.0050 < \epsilon$

3. Final Conclusion: Since $|x_3 - x_2| < 0.01$, we stop the iterations. The approximate root is:

$$\alpha \approx \mathbf{1.326}$$

5.5 Advantages and Disadvantages of the Fixed-Point Method

Like all numerical techniques, the success of the Fixed-Point method depends on the nature of the function and the choice of the transformation $g(x)$.

Advantages of Fixed-Point Method

- ✓ **Implementation Simplicity:** The method is very easy to program and requires only one initial guess x_0.
- ✓ **Flexibility:** A single equation $f(x) = 0$ can be transformed into multiple forms of $x = g(x)$, allowing different approaches to find the root.
- ✓ **Theoretical Foundation:** It provides a strong mathematical basis for understanding more complex iterative methods (like Newton's method).

Disadvantages of Fixed-Point Method

- × **Conditional Convergence:** Convergence is not guaranteed. It strictly requires the condition $|g'(x)| < 1$ near the root.
- × **Slow Convergence:** It generally has **Linear Convergence**, meaning it may take many iterations to reach a high-precision solution.
- × **Choice of $g(x)$:** Finding a suitable transformation $g(x)$ that satisfies the contraction mapping theorem can sometimes be difficult.
- × **Local Nature:** If x_0 is chosen poorly, the sequence may diverge even if a fixed point exists.

6 Newton-Raphson Method (Tangent Method)

6.1 Principle

Definition 6.1 Newton-Raphson Iteration

Let f be a function of class $C^2([a,b])$ such that $f'(x) \neq 0$ for all $x \in [a,b]$. Suppose the equation $f(x) = 0$ has a single solution $\alpha \in [a,b]$.
The idea is to approximate the curve $y = f(x)$ by its **tangent** at a point x_n. The next approximation x_{n+1} is the intersection of this tangent with the x-axis. The equation of the tangent at $(x_0, f(x_0))$ is:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Setting $y = 0$ and solving for x (which we call x_1), we get:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general, the sequence is generated by the recurrence relation:

$$\begin{cases} x_0 \in [a,b] \text{ (Initial guess)} \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \end{cases} \tag{16}$$

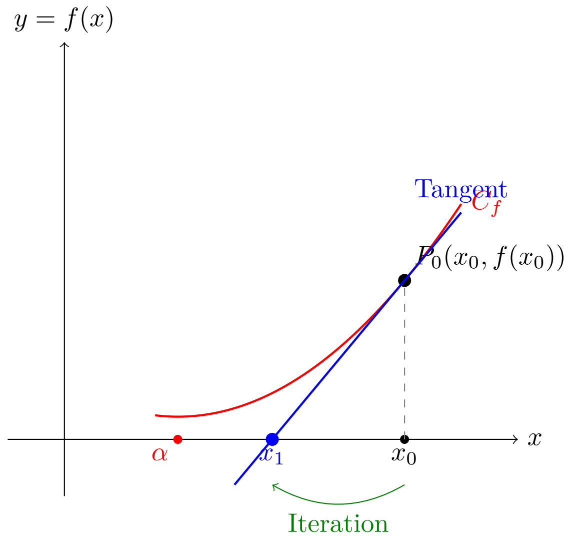


Figure 3: Geometric interpretation of the Newton-Raphson method.

6.2 Convergence Analysis

Proposition 6.1 Global Convergence Theorem

et f be a function of class C^2 on $[a, b]$ satisfying the following conditions:

i) $f(a) \times f(b) < 0$ (Root existence).

ii) $f'(x) \neq 0$ for all $x \in [a, b]$ (No critical points).

iii) $f''(x)$ maintains a constant sign on $[a, b]$ (No inflection points).

If we choose an initial guess $x_0 \in [a, b]$ such that $f(x_0) \cdot f''(x_0) \geq 0$ (**Fourier Condition**), then the sequence (x_n) converges quadratically to the unique solution α.

6.3 Numerical Application: Newton-Raphson Method

Example 6.1 Solving $x^3 - x - 1 = 0$

Find the positive root of $f(x) = x^3 - x - 1 = 0$ in the interval $[1, 2]$ using the Newton-Raphson method with a precision of $\epsilon = 0.01$.

Solution

1. Verification of Conditions: Before applying the method, we must check the following conditions on $[1, 2]$:

- **Existence:** $f(1) = -1$ and $f(2) = 5$. Since $f(1) \cdot f(2) < 0$, a root exists.
- **Monotonicity:** $f'(x) = 3x^2 - 1$. For $x \in [1, 2]$, $f'(x) \geq 2 > 0$. The function is strictly increasing, so the root is unique.
- **Concavity:** $f''(x) = 6x$. For $x \in [1, 2]$, $f''(x) > 0$. The sign of f'' is constant.
- **Choice of x_0:** We need $f(x_0) \cdot f''(x_0) > 0$. For $x_0 = 1.5$: $f(1.5) = 0.875 > 0$ and $f''(1.5) = 9 > 0$. The condition is satisfied.

Solution

2. Iterative Calculation ($x_0 = 1.5$): The recurrence formula is: $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

- **Iteration 1:** $x_1 = 1.5 - \frac{0.875}{5.75} \approx 1.3478$
  Error: $|x_1 - x_0| = 0.1522 > \epsilon$
- **Iteration 2:** $x_2 = 1.3478 - \frac{f(1.3478)}{f'(1.3478)} \approx 1.3252$
  Error: $|x_2 - x_1| = 0.0226 > \epsilon$
- **Iteration 3:** $x_3 = 1.3252 - \frac{f(1.3252)}{f'(1.3252)} \approx 1.3247$
  Error: $|x_3 - x_2| = 0.0005 < \epsilon$

3. Final Conclusion: Since $|x_3 - x_2| < 0.01$, the approximate root is:

$$\alpha \approx \mathbf{1.325}$$

6.4 Advantages and Disadvantages of the Newton-Raphson Method

Advantages of Newton-Raphson Method

- ✓ **Rapid Convergence:**
- ✓ **High Efficiency:**
- ✓ **Simplicity of Use:**

Disadvantages of Newton-Raphson Method

- × **Derivative Requirement:** It requires the calculation of the first derivative $f'(x)$. In some practical engineering problems, $f'(x)$ might be difficult or impossible to find analytically.
- × **Sensitivity to x_0:** If the initial guess x_0 is far from the true root α, the method may diverge or converge to a different root.
- × **Division by Zero:** If at any point $f'(x_n) = 0$ (a horizontal tangent), the formula becomes undefined, and the method fails.
- × **Local Convergence:** Unlike the Bisection method, it does not guarantee global convergence unless the initial guess is sufficiently close to the root.