Mohamed Boudiaf University - M'sila
Faculty of Science and Technology
Department of Civil Engineering

Numerical Methods Course
Chapter 6: Direct Methods for Solving Linear Systems

Merini Abdelaziz

Contents

1 Introduction

In this chapter, we focus on the numerical solution of a linear system:

$$Ax = b$$

Where:

- A is a square matrix of order n, assumed to be **invertible** ($\det(A) \neq 0$).
- b is the constant vector (right-hand side).
- x is the vector of unknowns.

Direct methods solve the system in a finite number of steps by transforming or factorizing the matrix A. The main methods discussed in this chapter are:

- **Gauss Elimination (Pivoting)**
- **LU Factorization**
- **Cholesky Factorization**
- **Thomas Algorithm (TDMA)**

Remark 1.1: Scope of Direct Methods

These methods are specifically used for **dense matrices** and systems of relatively small size (where n is not very large).

2 Definitions

Definition 2.1: Linear System

A linear system of n equations with n unknowns $x_1, \dots, x_n$ is a set of equations of the form:

$$(S) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Remark 2.1: Matrix Form

The system (S) can be written in matrix notation as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Which is equivalent to: $Ax = b$.

2.1 Triangular Matrices

2.1.1 Upper Triangular Matrix

Definition 2.2: Upper Triangular Matrix

A square matrix A is said to be **Upper Triangular** if all its elements below the main diagonal are zero:

$$a_{ij} = 0, \quad \forall i > j$$

Example 2.1: Upper Triangular Examples

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

2.1.2 Lower Triangular Matrix

Definition 2.3: Lower Triangular Matrix

A square matrix A is said to be **Lower Triangular** if all its elements above the main diagonal are zero:

$$a_{ij} = 0, \quad \forall i < j$$

Example 2.2: Lower Triangular Examples

$$C = \begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -5 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Remark 2.2: Properties of Triangular Matrices

1. A matrix that is both upper and lower triangular is called a **Diagonal Matrix**.
2. The determinant of a triangular matrix is the product of its diagonal elements:

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

2.1.3 Transpose of a Matrix

Definition 2.4: Matrix Transpose

The transpose of a matrix A of size (n, p), denoted by A^T, is a matrix of size (p, n) obtained by interchanging its rows and columns. Mathematically, if $A = (a_{ij})$, then $A^T = (a_{ji})$.

Example 2.3: Transpose Operations

Let A and B be two matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 5 \\ 6 & -2 & 3 \end{bmatrix} \implies B^T = \begin{bmatrix} 1 & 6 \\ 0 & -2 \\ 5 & 3 \end{bmatrix}$$

3 Gaussian Elimination Method

The Gaussian elimination method, also known as the **row reduction** or **pivot method**, is an algorithm used to find the set of solutions for a linear system $Ax = b$, where A is an $n \times n$ invertible matrix.

3.1 Principle of the Method

The principle consists of transforming the matrix A into an **upper triangular matrix** through forward elimination.

$$[A \mid b] \xrightarrow{\text{Transformation}} [\tilde{A} \mid \tilde{b}]$$

where $\tilde{A}$ is an upper triangular matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{a}_{nn} & \tilde{b}_n \end{array}\right]$$

After reaching the triangular form, the system $\tilde{A}x = \tilde{b}$ is solved using **back substitution**.

3.2 Steps of the Method

Row Update Rule

At step k ($1 \leq k \leq n-1$), if the pivot $a_{kk}^{(k)} \neq 0$ (otherwise, a row permutation is performed), the rows L_i for $k+1 \leq i \leq n$ are updated as:

$$L_i^{(k+1)} \leftarrow L_i^{(k)} - \left(\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \right) L_k^{(k)}$$

Remark: Calculations

1. The determinant of A can be calculated as: $\det(A) = (-1)^p \prod_{i=1}^{n} a_{ii}^{(i)}$, where p is the number of row permutations.

2. The method without row permutations is called "Ordinary Gaussian Elimination".

3.3 Numerical Example

Example: Solving System (S)

Solve the system:

$$(S)\begin{cases} 2x_1 + x_2 + 0x_3 + 4x_4 = 2 \\ -4x_1 - 2x_2 + 3x_3 - 7x_4 = -9 \\ 4x_1 + x_2 - 2x_3 + 8x_4 = 2 \\ 0x_1 - 3x_2 - 12x_3 - 4x_4 = 2 \end{cases}$$

Step 1: Forward elimination with pivot $a_{11} = 2$.

$$\left[\begin{array}{cccc|c} [2] & 1 & 0 & 4 & 2 \\ -4 & -2 & 3 & -7 & -9 \\ 4 & 1 & -2 & 8 & 2 \\ 0 & -3 & -12 & -4 & 2 \end{array}\right] \begin{array}{l} L_2 \leftarrow L_2 + 2L_1 \\ L_3 \leftarrow L_3 - 2L_1 \end{array} \rightarrow \left[\begin{array}{cccc|c} [2] & 1 & 0 & 4 & 2 \\ 0 & [0] & 3 & 1 & -5 \\ 0 & -1 & -2 & 0 & -2 \\ 0 & -3 & -12 & -4 & 2 \end{array}\right]$$

Step 2: Since $a_{22} = 0$, we swap $L_2 \leftrightarrow L_3$:

$$\xrightarrow{L_2 \leftrightarrow L_3} \left[\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 2 \\ 0 & [-1] & -2 & 0 & -2 \\ 0 & 0 & 3 & 1 & -5 \\ 0 & -3 & -12 & -4 & 2 \end{array}\right] L_4 \leftarrow L_4 - 3L_2 \rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 2 \\ 0 & -1 & -2 & 0 & -2 \\ 0 & 0 & 3 & 1 & -5 \\ 0 & 0 & -6 & -4 & 8 \end{array}\right]$$

Step 3: Final elimination using $a_{33} = 3$. After back substitution, we find: $\mathbf{x} = [\mathbf{-4}, \mathbf{6}, \mathbf{-2}, \mathbf{1}]^{\mathbf{T}}$.

4 LU Decomposition Method

4.1 Principle of LU Factorization

The LU decomposition (or factorization) is a method that factors a square matrix A into the product of a lower triangular matrix L and an upper triangular matrix U.

Definition 4.1: LU Decomposition

Let A be a square matrix of order n. The LU decomposition of A consists of finding two matrices L and U such that:

$$A = LU$$

Where:

- L is a **unit lower triangular** matrix (all diagonal elements are 1).
- U is an **upper triangular** matrix.

Remark 4.1: Solving the System

Once A is factorized as $A = LU$, the system $Ax = b$ becomes $L(Ux) = b$. This is solved in two stages:

$$Ax = b \iff L(Ux) = b \iff \begin{cases} Ly = b & (1) \\ Ux = y & (2) \end{cases}$$

4.2 Existence and Uniqueness

Definition 4.2: Leading Principal Sub-matrices

The leading principal sub-matrices of A, denoted by $A_{[k]}$, are the $k \times k$ sub-matrices formed by the first k rows and k columns of A.

Existence Theorem (General Case)

A square matrix A of order n admits a unique LU factorization if and only if all its leading principal sub-matrices are non-singular:

$$\det(A_{[k]}) \neq 0, \quad \forall k = 1, \ldots, n$$

Example 4.1

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det(A_{[1]}) = 1 \neq 0$ and $\det(A_{[2]}) = -2 \neq 0$. Thus, LU exists.

Theorem for Diagonally Dominant Matrices

If A is a **strictly diagonally dominant** matrix (by rows or columns), then it admits a unique LU factorization.

Example 4.2

For $A = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$, row 1: $5 > |2|$, row 2: $4 > |1|$. Since it is diagonally dominant, LU factorization is guaranteed without checking determinants.

4.3 Numerical Application

Example 4.3: Complete Resolution by LU - Part 1: Factorization

Consider the linear system $(S) \iff Ax = b$:

$$\begin{bmatrix} 5 & 3 & 1 \\ 10 & 8 & 1 \\ 0 & -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \\ 161 \end{bmatrix}$$

1. Verification of Existence (Leading Principal Minors): To ensure a unique LU factorization exists, we check the determinants:

- $\det A_{[1]} = |5| = 5 \neq 0.$, $\det A_{[2]} = \det \begin{bmatrix} 5 & 3 \\ 10 & 8 \end{bmatrix} = (5 \times 8) - (10 \times 3) = 10 \neq 0.$ and $\det A_{[3]} = \det(A) = 5(40+6) - 10(15+6) = 230 - 210 = 20 \neq 0.$

Since all leading principal minors are non-zero, A has a unique decomposition $A = LU$.

Example 4.4: Complete Resolution by LU - Part 2: System Solution

2. Factorization Step (By Identification): We set $A = LU$:

$$\begin{bmatrix} 5 & 3 & 1 \\ 10 & 8 & 1 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

By identification, we solve for the unknowns row by row:

- **Row 1:** $u_{11} = 5, \quad u_{12} = 3, \quad u_{13} = 1.$
- **Row 2:**
  - $l_{21}(5) = 10 \implies l_{21} = 2.$
  - $l_{21}(3) + u_{22} = 8 \implies 2(3) + u_{22} = 8 \implies u_{22} = 2.$
  - $l_{21}(1) + u_{23} = 1 \implies 2(1) + u_{23} = 1 \implies u_{23} = -1.$
- **Row 3:**
  - $l_{31}(5) = 0 \implies l_{31} = 0.$
  - $l_{31}(3) + l_{32}(2) = -6 \implies 0 + 2l_{32} = -6 \implies l_{32} = -3.$
  - $l_{31}(1) + l_{32}(-1) + u_{33} = 5 \implies 0 + (-3)(-1) + u_{33} = 5 \implies u_{33} = 2.$

Result: $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$

Using the matrices L and U calculated in the previous step:

3. Forward Substitution (Solving $Ly = b$):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \\ 161 \end{bmatrix}$$

From the first equation: $y_1 = 12$.
From the second: $2(12) + y_2 = -13 \implies y_2 = -13 - 24 = -37$.
From the third: $-3(-37) + y_3 = 161 \implies 111 + y_3 = 161 \implies y_3 = 50$.
So, $y = [12, -37, 50]^T$.

4. Backward Substitution (Solving $Ux = y$):

$$\begin{bmatrix} 5 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -37 \\ 50 \end{bmatrix}$$

From the third equation: $2x_3 = 50 \implies x_3 = 25$.
From the second: $2x_2 - 25 = -37 \implies 2x_2 = -12 \implies x_2 = -6$.
From the first: $5x_1 + 3(-6) + 25 = 12 \implies 5x_1 - 18 + 25 = 12 \implies 5x_1 + 7 = 12 \implies x_1 = 1$.

Final Solution: The solution vector for the system (S) is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

Remark 4.2: Determinant Calculation

The determinant of A can be calculated from the LU factorization as:

$$\det(A) = \det(L) \times \det(U) = 1 \times \prod_{i=1}^{n} u_{ii}$$

For the previous example: $\det(A) = 5 \times 2 \times 2 = 20$.

5 Cholesky Factorization Method (MM^T)

5.1 Principle of Cholesky Method

The Cholesky method is a specialized version of LU decomposition for symmetric, positive-definite matrices.

Method Steps

Consider the linear system $Ax = b$, where A is a **Symmetric Positive Definite (SPD)** matrix. The resolution proceeds as follows:

1. **Decomposition:** Factor A into $A = MM^T$, where M is a lower triangular matrix with positive diagonal elements.
2. **System Resolution:**

$$Ax = b \iff M(M^T x) = b \iff \begin{cases} My = b & \textit{(1)} \\ M^T x = y & \textit{(2)} \end{cases}$$

5.2 Numerical Application: Cholesky Method

Example 5.1: Cholesky Resolution (Part 1: Factorization)

Consider the linear system $Ax = b$ where $b = [28, 96, -985]^T$ and:

$$A = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix}$$

i) Symmetry Verification: $A^T = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} = A.$

ii) Positive Definiteness Verification:

- $\det A_{[1]} = 1 > 0$
- $\det A_{[2]} = \det \begin{pmatrix} 1 & -3 \\ -3 & 13 \end{pmatrix} = 13 - 9 = 4 > 0$
- $\det A_{[3]} = \det(A) = 1(533 - 64) + 3(-123) = 100 > 0$

iii) Matrix Factorization ($A = MM^T$): We set $M = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ such that $A = MM^T$:

$$\begin{pmatrix} 1 & -3 & 0 \\ -3 & 13 & -8 \\ 0 & -8 & 41 \end{pmatrix} = \begin{pmatrix} m_{11}^2 & m_{11}m_{21} & m_{11}m_{31} \\ m_{11}m_{21} & m_{21}^2 + m_{22}^2 & m_{21}m_{31} + m_{22}m_{32} \\ m_{11}m_{31} & m_{21}m_{31} + m_{22}m_{32} & m_{31}^2 + m_{32}^2 + m_{33}^2 \end{pmatrix}$$

By identifying the coefficients:

$$\begin{cases} m_{11} = \sqrt{1} = 1 \\ m_{21} = -3/1 = -3 \\ m_{31} = 0/1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} m_{22} = \sqrt{13 - 9} = 2 \\ m_{32} = -8/2 = -4 \\ m_{33} = \sqrt{41 - 16} = 5 \end{cases}$$

Result: $M = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & -4 & 5 \end{pmatrix}, \quad M^T = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 5 \end{pmatrix}.$

Example 5.1: Cholesky Resolution (Part 2: Solving)

iv) System Resolution: The system $Ax = b \iff M(M^T x) = b$. We solve in two steps:

1. **Forward Substitution ($My = b$):**

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & -4 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 96 \\ -985 \end{pmatrix} \iff \begin{cases} y_1 = 28 \\ -3y_1 + 2y_2 = 96 \Rightarrow y_2 = 90 \\ -4y_2 + 5y_3 = -985 \Rightarrow y_3 = -125 \end{cases}$$

2. **Backward Substitution ($M^T x = y$):**

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 90 \\ -125 \end{pmatrix} \iff \begin{cases} 5x_3 = -125 \Rightarrow x_3 = -25 \\ 2x_2 - 4x_3 = 90 \Rightarrow x_2 = -5 \\ x_1 - 3x_2 = 28 \Rightarrow x_1 = 13 \end{cases}$$

Final Solution: The solution vector is $\mathbf{x} = [\mathbf{13}, -\mathbf{5}, -\mathbf{25}]^{\mathbf{T}}$.
v) Determinant Calculation: Using the diagonal elements of M:

$$\det(A) = (m_{11} \times m_{22} \times m_{33})^2 = (1 \times 2 \times 5)^2 = 100$$

6 LU Factorization of a Tridiagonal Matrix (THOMAS Algorithm)

Definition 6.1: Tridiagonal Matrix

A square matrix A of order n is said to be **tridiagonal** if all its non-zero coefficients are located only on the main diagonal (a_i), the super-diagonal (c_i), and the sub-diagonal (e_i):

$$A = \begin{pmatrix} a_1 & c_1 & 0 & \dots & 0 \\ e_2 & a_2 & c_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \dots & 0 & e_n & a_n \end{pmatrix}$$

6.1 Principle of Thomas Algorithm

The algorithm factors A into two bidiagonal matrices L and U:

$$L = \begin{pmatrix} 1 & & \\ \beta_2 & 1 & \\ & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} \alpha_1 & c_1 & \\ & \alpha_2 & c_2 \\ & & \ddots \end{pmatrix}$$

Recurrence and Resolution Formulas

1. Factorization Step:

$$\alpha_1 = a_1, \quad \beta_i = \frac{e_i}{\alpha_{i-1}}, \quad \alpha_i = a_i - \beta_i c_{i-1} \quad (i = 2, \dots, n)$$

2. Forward Substitution ($Ly = b$):

$$y_1 = b_1, \quad y_i = b_i - \beta_i y_{i-1}$$

3. Backward Substitution ($Ux = y$):

$$x_n = \frac{y_n}{\alpha_n}, \quad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i}$$

6.2 Existence and Stability Conditions

To guarantee that the Thomas algorithm can be performed successfully, the coefficients of the original matrix A should satisfy certain properties.

Conditions for matrix diagonals A

The factorization is guaranteed to exist and be numerically stable if:

1. **Initial Condition:** The first diagonal element must be non-zero: $a_1 \neq 0$.
2. **Diagonal Dominance (Sufficient Condition):** The matrix A should be strictly diagonally dominant, meaning:

$$\begin{cases} |a_1| > |c_1| \\ |a_i| \geq |e_i| + |c_i|, \quad \text{for } i = 2, \dots, n-1 \\ |a_n| > |e_n| \end{cases}$$

6.3 Numerical Application: 4×4 Tridiagonal System

Example 6.1: Thomas Algorithm Resolution (Part 1)

Consider the system $Ax = b$ with:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 9 \\ 10 \\ 11 \end{pmatrix}$$

Step 1: LU Factorization

- $i = 1$**:** $\alpha_1 = a_1 = 1$.
- $i = 2$**:** $\beta_2 = 2/1 = 2, \quad \alpha_2 = 3 - (2)(1) = 1$.
- $i = 3$**:** $\beta_3 = 1/1 = 1, \quad \alpha_3 = 2 - (1)(1) = 1$.
- $i = 4$**:** $\beta_4 = 1/1 = 1, \quad \alpha_4 = 2 - (1)(1) = 1$.

The bidiagonal matrices are:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 6.1: Part 2 (Resolution)

Step 2: Forward Substitution ($Ly = b$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 10 \\ 11 \end{pmatrix} \implies \begin{cases} y_1 = 3 \\ y_2 = 9 - (2)(3) = 3 \\ y_3 = 10 - (1)(3) = 7 \\ y_4 = 11 - (1)(7) = 4 \end{cases}$$

Step 3: Backward Substitution ($Ux = y$)

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 7 \\ 4 \end{pmatrix} \implies \begin{cases} x_4 = 4/1 = 4 \\ x_3 = (7 - 1 \times 4)/1 = 3 \\ x_2 = (3 - 1 \times 3)/1 = 0 \\ x_1 = (3 - 1 \times 0)/1 = 3 \end{cases}$$

Final Solution: $\mathbf{x} = [\mathbf{3}, \mathbf{0}, \mathbf{3}, \mathbf{4}]^{\mathbf{T}}$.