

Solutions to the Directed Works Series n°6

Exercice n°1 :

We toss a fair coin 3 times. Let X be the number of heads obtained.

- 1) Find the sample space Ω of this random experiment.
- 2)) Find the probability distribution of X , its expectation $E(X)$ and its variance $V(X)$.
- 3) Calculate $P(X < 2)$ and $P(X \geq 2)$ and $P(1 < X < 2.5)$.
- 4)) Let $Y = 3X + 1$. deduce its expectation $E(Y)$ and its variance $V(Y)$.

Solution :

We toss a fair coin 3 times. Let X be the number of heads obtained.

1. The sample space is :

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}, \quad |\Omega| = 8.$$

2. The probability distribution of X is given by :

$$P(X = k) = \frac{|A_k|}{8}, \quad k = 0, 1, 2, 3$$

X	0	1	2	3
P	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expectation :

$$E(X) = \sum_{k=0}^3 k P(X = k) = \frac{3}{2}.$$

Variance :

$$E(X^2) = \sum_{k=0}^3 k^2 P(X = k) = 3, \quad V(X) = E(X^2) - [E(X)]^2 = \frac{3}{4}.$$

3. Required probabilities :

$$P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{2},$$

$$P(X \geq 2) = P(X = 2) + P(X = 3) = \frac{1}{2},$$

$$P(1 < X < 2.5) = P(X = 2) = \frac{3}{8}.$$

4. For $Y = 3X + 1$:

$$E(Y) = 3E(X) + 1 = \frac{11}{2}, \quad V(Y) = 9V(X) = \frac{27}{4}.$$

Exercice n°2 :

Let X be a random variable whose probability distribution is given by

X	4	5	6	7	8
P	0.15	0.35	0.10	0.25	0.15

- 1) Find and represent the distribution function of X .
- 2)) Calculate the following probabilities : $P(X \leq 7.5)$; $P(X > 8)$; $P(4 \leq X \leq 6.5)$; $P(5 < X \leq 6)$.
- 3) Determine the expectation and variance of X .
- 4)) Let $Z = \frac{1}{X}$. Find the distribution of Z , and its expectation $E(Z)$ and variance and its variance $V(Z)$.

Exercice n°2 :

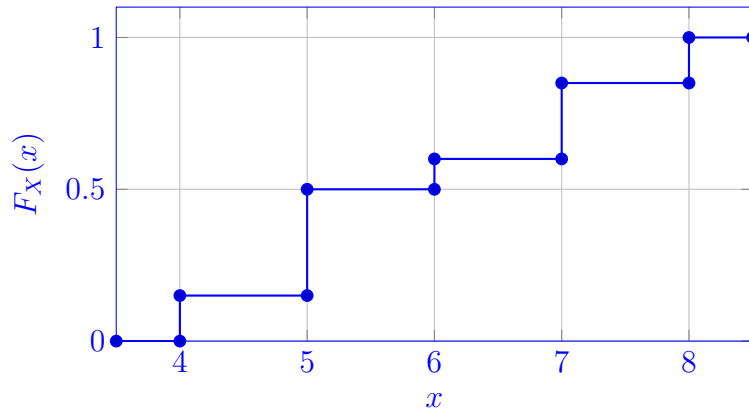
Solution :

Let X be a random variable whose probability distribution is given by :

X	4	5	6	7	8
$P(X)$	0.15	0.35	0.10	0.25	0.15

1. The distribution function $F_X(x) = P(X \leq x)$ is :

$$F_X(x) = \begin{cases} 0, & x < 4, \\ 0.15, & 4 \leq x < 5, \\ 0.50, & 5 \leq x < 6, \\ 0.60, & 6 \leq x < 7, \\ 0.85, & 7 \leq x < 8, \\ 1, & x \geq 8. \end{cases}$$



2. Required probabilities :

$$P(X \leq 7.5) = P(X \leq 7) = 0.85,$$

$$P(X > 8) = 0,$$

$$P(4 \leq X \leq 6.5) = P(X = 4) + P(X = 5) + P(X = 6) = 0.60,$$

$$P(5 < X \leq 6) = P(X = 6) = 0.10.$$

3. Expectation and variance of X :

$$E(X) = 4(0.15) + 5(0.35) + 6(0.10) + 7(0.25) + 8(0.15) = 5.9,$$

$$E(X^2) = 4^2(0.15) + 5^2(0.35) + 6^2(0.10) + 7^2(0.25) + 8^2(0.15) = 36.6,$$

$$V(X) = E(X^2) - [E(X)]^2 = 36.6 - (5.9)^2 = 36.6 - 34.81 = 1.79.$$

4. For $Z = \frac{1}{X}$, the distribution is :

Z	$P(Z)$
1/4	0.15
1/5	0.35
1/6	0.10
1/7	0.25
1/8	0.15

Expectation :

$$E(Z) = 0.17863 \text{ (approx).}$$

Second moment :

$$E(Z^2) \approx 0.03360.$$

Variance :

$$V(Z) = E(Z^2) - [E(Z)]^2 \approx 0.00170.$$

Exercise n°3 : We consider a biased cubic die whose faces are numbered from 1 to 6, and let X be the random variable given by the number on the top face. We assume that $\exists a \in \mathbb{R}^+$ such that

$$P(X = k) = ka, \quad k = 1, 2, 3, 4, 5, 6.$$

1. Determine the probability distribution of X , and compute its expectation.
2. Let $Y = \frac{1}{X}$. Determine the distribution of Y , and compute its expectation and variance.

Solution :

1. Distribution of X and its expectation

Normalization gives :

$$\sum_{k=1}^6 \mathbb{P}(X = k) = \sum_{k=1}^6 ka = a \sum_{k=1}^6 k = a \cdot 21 = 1 \quad \Rightarrow \quad a = \frac{1}{21}.$$

Hence,

$$\mathbb{P}(X = k) = \frac{k}{21}, \quad k = 1, \dots, 6.$$

k	$\mathbb{P}(X = k)$
1	$\frac{1}{21}$
2	$\frac{2}{21}$
3	$\frac{3}{21}$
4	$\frac{4}{21}$
5	$\frac{5}{21}$
6	$\frac{6}{21}$

Expectation :

$$E(X) = \sum_{k=1}^6 k \cdot \frac{k}{21} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{21} = \frac{91}{21} = \frac{13}{3}.$$

Second moment :

$$E(X^2) = \sum_{k=1}^6 k^2 \cdot \frac{k}{21} = \frac{\sum_{k=1}^6 k^3}{21} = \frac{(1 + 2 + \dots + 6)^2}{21} = \frac{21^2}{21} = 21.$$

Variance :

$$V(X) = E(X^2) - [E(X)]^2 = 21 - \left(\frac{13}{3}\right)^2 = \frac{20}{9}.$$

2. Distribution of $Y = \frac{1}{X}$, expectation, and variance

Support :

$$Y \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}, \quad \mathbb{P}\left(Y = \frac{1}{k}\right) = \mathbb{P}(X = k) = \frac{k}{21}, \quad k = 1, \dots, 6.$$

Expectation :

$$E(Y) = \sum_{k=1}^6 \frac{1}{k} \cdot \frac{k}{21} = \sum_{k=1}^6 \frac{1}{21} = \frac{6}{21} = \frac{2}{7}.$$

Second moment :

$$E(Y^2) = \sum_{k=1}^6 \frac{1}{k^2} \cdot \frac{k}{21} = \frac{1}{21} \sum_{k=1}^6 \frac{1}{k} = \frac{H_6}{21} = \frac{\frac{147}{60}}{21} = \frac{49}{420} = \frac{7}{60}.$$

Variance :

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{7}{60} - \left(\frac{2}{7}\right)^2 = \frac{7}{60} - \frac{4}{49} = \frac{103}{2940}.$$