Mohamed Boudiaf University - M'sila **University Year : 2025/2026**
Faculty of Technology
Department of Civil Engineering-Department of Electrical Engineering
Module : Probabilities and Statistics

Solutions to the Directed Works Series n°4

Exercice n°1 :

In a tutorial group, 15% of Maths 1 marks were insufficient, 25% of Physics 1 marks were insufficient and 10% of students had insufficient marks in both modules.

a) A student has an insufficient mark in physics 1. Calculate the probability that he also has a mark that is insufficient in Maths 1.

b) A student has a mark that is insufficient in Maths 1. Calculate the probability that he also has a mark that is insufficient in Physics 1.

Solution :

Given : $P(M) = 0.15, \quad P(ph) = 0.25, \quad P(M \cap ph) = 0.10$

(a) We want $P_{ph}(M) = \frac{P(M \cap ph)}{P(ph)} \quad \Rightarrow \quad P_{ph}(M) = \frac{0.10}{0.25} = 0.4 = 40\%.$

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(b) We want $P_M(ph) = \frac{P(M \cap P)}{P(M)} \quad \Rightarrow \quad P_M(ph) = \frac{0.10}{0.15} = \frac{2}{3} \approx 0.6667 = 66.7\%.$

Exercice n°2 : A coin and a dice are tossed. Consider the following events :.

A = "Get heads" ; B = "Get a number < 3".

1) Determine the universe Ω of this random experiment.

2) Calculate $P(A), P(B), P(A \cap B)$, and $P_B(A)$.

3) Are events A and B independent ? (justify your answer).

4) Are events A and B incompatible ? (justify your answer).

Solution :

Exercise n°2 :

A coin and a die are tossed. Consider the following events : A = "Get heads" ; B = "Get a number < 3".

1) Determine the universe Ω :

The sample space is the set of all possible outcomes of tossing a coin and a die :

$$\Omega = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

Thus, $|\Omega| = 12$.

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2) Calculate probabilities :

- Event A = "Get heads" : $A = (H,1),(H,2),(H,3),(H,4),(H,5),(H,6)$ .Thus, $|A| = 12$.

$$P(A) = \frac{6}{12} = \frac{1}{2}.$$

- Event $B =$ "Get a number < 3" : $B = (H,1),(H,2),(T,1),(T,2$. Thus, $|B| = 4$

$$P(B) = \frac{4}{12} = \frac{4}{12} = \frac{1}{3}.$$

- Intersection $A \cap B =$ "Get heads and a number < 3" :

$$A \cap B = \{(H,1),(H,2)\}, \quad P(A \cap B) = \frac{2}{12} = \frac{1}{6}.$$

- Conditional probability

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}.$$

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3) Independence :

Events A and B are independent if :

$$P(A \cap B) = P(A) \cdot P(B).$$

Check :

$$P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

Since $P(A \cap B) = \frac{1}{6}$, the equality holds.

$$\Rightarrow \text{Events } A \text{ and } B \text{ are independent.}$$

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4) Incompatibility :

Events are incompatible if $P(A \cap B) = 0$.
Here, $P(A \cap B) = \frac{1}{6} \neq 0$.

$$\Rightarrow \text{Events } A \text{ and } B \text{ are not incompatible.}$$

Exercice n°3 :

Three machines, A, B, and C, produce $50\%, 30\%$, and 20% of the total number of parts manufactured in a factory, respectively. The percentages of defective parts produced by these machines are $3\%, 4\%$, and 5%. Take a part at random and calculate the probability :

a) a part is defective.

b) a defective part comes from A.

c) a non-defective part comes from C.

Solution :

Given :

$$P(A) = 0.5, \ P(B) = 0.3, \ P(C) = 0.2,$$

$$A \cap B = \varnothing,\ A \cap C = \varnothing,\ B \cap C = \varnothing, \qquad A \cup B \cup C = \Omega, \qquad P(A) + P(B) + P(C) = 1.$$

$\{A, B, C\}$ form a partition of Ω.
Let D = defective part. $\bar{D}$ = non-defective part.

$$P_A(D) = 0.03,\ P_B(D) = 0.04,\ P_C(D) = 0.05,$$

a) We are looking for P(D) :

$$\begin{aligned} P(D) &= P(A \cap D) + P(B \cap D) + \ldots + P(C \cap D) \\ &= P(A)\, P_A(D) + P(B)\, P_B(D) + P(C)\, P_C(D) \\ &= 0.5 \cdot 0.03 + 0.3 \cdot 0.04 + 0.2 \cdot 0.05 \\ &= 0.015 + 0.012 + 0.010 \\ &= 0.037. \end{aligned}$$

b) We are looking for $P_D(A)$

$$P_D(A) = \frac{P(A)\, P_A(D)}{P(D)} = \frac{0.5 \cdot 0.03}{0.037} = \frac{0.015}{0.037} \approx 0.405.$$

c) We are looking for $P_{\bar{D}}(C)$

$$P(\bar{D}) = 1 - P(D) = 1 - 0.037 = 0.963, \qquad P_C(\bar{D}) = 1 - P_C(D) = 1 - 0.05 = 0.95,$$

$$\begin{aligned} P_{\bar{D}}(C &= \frac{P(C)\, P_C(\bar{D})}{P(\bar{D})} \\ &= \frac{P(C)\, P_C(\bar{D})}{P(\bar{D})} \\ &\frac{0.2 \times 0.95}{0.963} \\ &= \frac{0.19}{0.963} \approx 0.197. \end{aligned}$$

$$\boxed{\text{Results : } \quad P(D) = 0.037, \quad P_D(A) \approx 0.405, \quad P_{\bar{D}}(C) \approx 0.197.}$$

<u>Exercice n°4 :</u>
Let $A, B, C \subset \Omega$,such that $P(B) > 0$. Show that

a) $P_B(\phi) = 0; \quad P_B(\Omega) = 1.$

b) $P(\bar{A}/B) = 1 - P_B(A)$

c) $P_B(C \cup A) = P_B(C) + P_B(A) - P_B(C \cap A)$

<u>Solution :</u>

$A, B, C \subset \Omega \qquad P(B) > 0.$

$$\textbf{Reminder} \quad \forall E \subset \Omega, \quad P_B(E) := P(E \,|\, B) = \frac{P(E \cap B)}{P(B)}.$$

We will use the properties of probabilities : $P(\varnothing) = 0,\ P(\Omega) = 1,\ P(X \cup Y) = P(X) + P(Y) - P(X \cap Y).$

(a)

$P_B(\varnothing) = \dfrac{P(\varnothing \cap B)}{P(B)} = \frac{P(\varnothing)}{P(B)} = \frac{0}{P(B)} = 0.$

$P_B(\Omega) = \dfrac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

(b)

$$: \quad P_B(\overline{A}) = \frac{P(\overline{A} \cap B)}{P(B)}.$$

$$B = (A \cap B) \dot{\cup} (\overline{A} \cap B) \text{ and } \varnothing = (A \cap B) \dot{\cap} (\overline{A} \cap B) \Rightarrow\ P(B) = P(A \cap B) + P(\overline{A} \cap B).$$

$$\Rightarrow\ P(\overline{A} \cap B) = P(B) - P(A \cap B).$$

$$therefore\ P_B(\overline{A}) = \frac{P(B) - P(A \cap B)}{P(B)} = 1 - \frac{P(A \cap B)}{P(B)} = 1 - P_B(A).$$

(c)

$$P_B(C \cup A) = P_B(C) + P_B(A) - P_B(C \cap A).$$

$$P_B(C \cup A) = \frac{P\big((C \cup A) \cap B\big)}{P(B)}.$$

We use the distribution of intersections on the union : $\quad (C \cup A) \cap B = (C \cap B) \cup (A \cap B).$

$$P\big((C \cap B) \cup (A \cap B)\big) = P(C \cap B) + P(A \cap B) - P\big((C \cap B) \cap (A \cap B)\big).$$

$$(C \cap B) \cap (A \cap B) = (C \cap A) \cap B.$$

$$\Rightarrow\ \frac{P\big((C \cap B) \cup (A \cap B)\big)}{P(B)} = \frac{P(C \cap B)}{P(B)} + \frac{P(A \cap B)}{P(B)} - \frac{P((C \cap A) \cap B)}{P(B)}.$$

$$P_B(C \cup A) = P_B(C) + P_B(A) - P_B(C \cap A).$$

Exercice n°5 :

Let $A, B \subset \Omega$, . suppose that A and B independent. Show that

a) $\bar{A}$ and B are independent.

b) A and $\bar{B}$ are independent.

c) $\bar{A}$ and $\bar{B}$ are independent.

Solution : Let $A, B \subset \Omega$. Suppose that A and B are independent, i.e., $P(A \cap B) = P(A)\,P(B)$. Show that :

(a) $\overline{A}$ and B are independent.

(b) A and $\overline{B}$ are independent.

(c) $\overline{A}$ and $\overline{B}$ are independent.

Solution.

(a) Independence of $\overline{A}$ and B :

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)P(B)$$

$$= P(B)\big(1 - P(A)\big)$$

$$= P(B)\,P(\overline{A}).$$

Hence, $\overline{A}$ and B are independent.

(b) Independence of A and $\overline{B}$:

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A)P(B)$$

$$= P(A)\big(1 - P(B)\big)$$

$$= P(A)\,P(\overline{B}).$$

Hence, A and $\overline{B}$ are independent.

(c) Independence of $\overline{A}$ and $\overline{B}$:

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$= 1 - \big(P(A) + P(B) - P(A \cap B)\big)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= \big(1 - P(A)\big)\big(1 - P(B)\big)$$

$$= P(\overline{A})\,P(\overline{B}).$$

Hence, $\overline{A}$ and $\overline{B}$ are independent.

Conclusion : If A and B are independent, then complements preserve independence as shown in (a), (b), and (c).

Module manager. Merini.A