Mohamed Boudiaf University - M'sila **University Year : 2025/2026**
Faculty of Technology
Department of Civil Engineering
Module : Probability and Statistics

Correcting the final exam

Course Question

Let Ω be a sample space and P a probability an Ω.

1) State the axioms of probability.

2) Let $A, B \subset \Omega$. Show that :

$$P\left(B \cap \bar{A}\right) = P(B) - P(A \cap B)$$

Solution :

1) The axioms of probability are :

i) **Non-negativity :** For all $A \subseteq \Omega$, $P(A) \geq 0$. (0.5 mark)

ii) **Normalization :** $P(\Omega) = 1$. (0.5 mark)

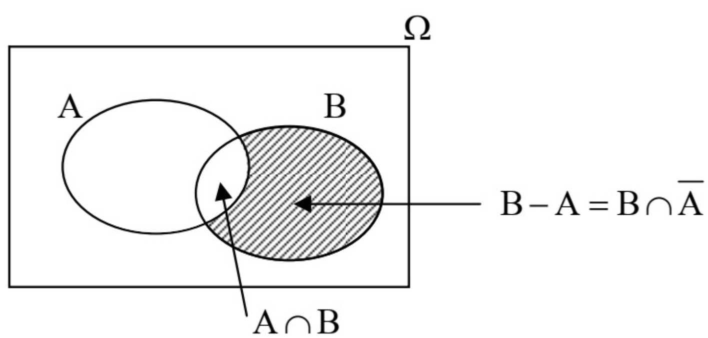
iii) **Additivity :** If $A \cap B = \varnothing$, then (0.5 mark)

$$P(A \cup B) = P(A) + P(B). \quad (0.5\ mark)$$

2) Proof of the identity :

$\left(B \cap \bar{A}\right) \cup (B \cap A) = B$ and $\left(B \cap \bar{A}\right) \cap (B \cap A) = \phi$, so (0.5 mark)

$$\begin{aligned} P(B) &= P\left(\left(B \cap \bar{A}\right) \cup (B \cap A)\right) \quad (0.5\ mark) \\ &= P\left(B \cap \bar{A}\right) + P(B \cap A) \quad (0.5\ mark) \text{ (By the axiom 3)} \\ &\Rightarrow P\left(B \cap \bar{A}\right) = P(B) - P(B \cap A) \quad (0.5\ mark) \end{aligned}$$



Exercise n°1 : (08 marks)

In a soil mechanics study, the settlement rate $S(t)$ of a clay layer under a foundation was measured over 28 days :

Time t (days)	4	8	12	16	20	24	28
Settlement Rate $S(t)$ (mm/day)	12.00	8.94	6.66	4.97	3.70	2.76	2.06

a) Draw the scatter plot (t, S) in an orthogonal frame :
Horizontal axis : $1\,\text{cm} \rightarrow 4\,\text{days}$. Vertical axis : $1\,\text{cm} \rightarrow 1\,\text{mm/day}$.

b) Is the relationship between S and t linear ? Justify.

c) Let $Z = \ln(S)$. Complete the table (round results to 3 decimal places) :

Time t (days)	4	8	12	16	20	24	28
$Z = \ln(S)$	2.485	2.191					

d) Find the equation of the regression line from z to t, using the method of least squares.

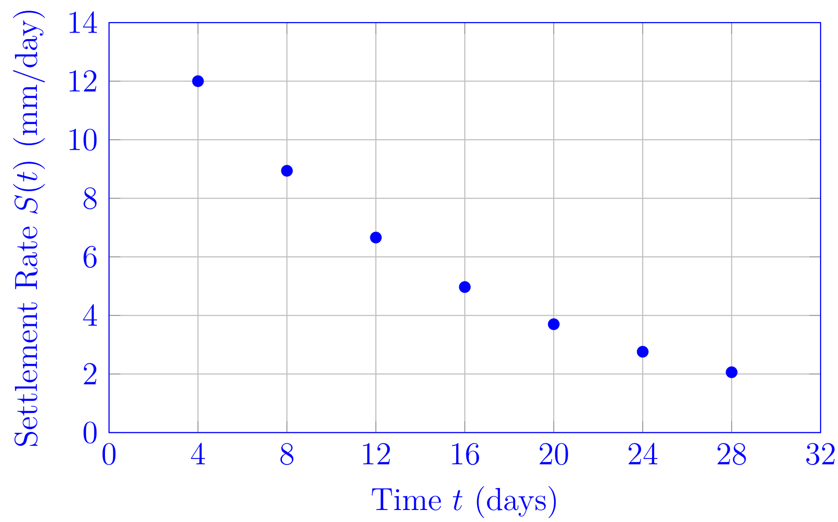
e) Deduce the equation of the settlement rate S in terms of t (in the form $S = Ae^{Bt}$).

f) Estimate the settlement rate S when $t = 40$ days.

Solution :

Solution :

a) Scatter Plot : (1.5 marks)



b) The relationship between S and t is **not linear** because the points on the scatter plot follow a curve and do not align on a single straight line. (0.5 mark)

c) Completion of the table ($Z = \ln S$) : (1.5 marks)

Time t (days)	4	8	12	16	20	24	28
$Z = \ln(S)$	2.485	2.191	**1.896**	**1.603**	**1.308**	**1.015**	**0.723**

d) Regression Line equation ($Z = at + b$) : Using the least squares method :

- Mean $\bar{t} = \frac{\sum t_i}{7} = 16$ (0.25 mark)
- Mean $\bar{Z} = \frac{\sum Z_i}{7} \approx 1.603$ (0.25 mark)
- Variance $V(t) = \frac{1}{7}\sum t_i^2 - \bar{t}^2 = 320 - 256 = 64$ (0.5 mark)
- $Cov(t, Z) = \frac{1}{7}\sum t_i Z_i - \bar{t}\bar{Z} \approx -4.697$ (0.5 mark)
- Slope $a = \frac{Cov(t,Z)}{V(t)} \approx -0.0734$ (0.5 mark)
- Intercept $b = \bar{Z} - a\bar{t} \approx 1.603 - (-0.0734 \times 16) \approx 2.777$ (0.5 mark)

The regression line equation is : $\mathbf{Z = -0.073t + 2.777}$ (0.5 mark)

e) Equation of S in terms of t ($S = Ae^{Bt}$) : Since $Z = \ln(S)$, then $S = e^Z = e^{-0.073t+2.777}$:

$$S(t) = e^{2.777} \cdot e^{-0.073t}$$

Calculating $A = e^{2.777} \approx 16.07$:

$$\mathbf{S(t) = 16.07e^{-0.073t}} (1 mark)$$

f) Estimation for $t = 40$ days : Substitute $t = 40$ into the equation :

$$S(40) = 16.07e^{-0.073\times 40} = 16.07e^{-2.92}$$

$$S(40) \approx 16.07 \times 0.0539 \approx \mathbf{0.866} \text{ mm/day}$$

(0.5 mark)

Exercise n°3 (3 marks)

A survey of soil samples from a highway project gives the following information :
40% of samples have High Clay Content (H), 18% of samples have Low Load-Bearing Capacity (L) and 10% of samples have both High Clay and Low Load-Bearing Capacity.

Express the percentage of samples :

a) Having high clay content or low load-bearing capacity.

b) Not having high clay content.

c) Having neither high clay content nor low load-bearing capacity.

d) Having low load-bearing capacity but not high clay content

Exercise n°2 (5 marks)

In a structural health monitoring study, let X be the number of micro-cracks detected in a 1-meter section of a concrete beam. The probability distribution function (PDF) of X is given by the following table :

x_i	0	1	2	3	4
$P(X = x_i)$	0.4	0.3	c	0.1	0.05

1. Determine the value of the constant c.
2. Determine the expression of the Cumulative Distribution Function (CDF), $F(x)$.
3. Calculate the probability that at least 2 cracks are detected in the section.
4. Calculate the expected number of cracks $E(X)$ and the variance $V(X)$.
5. If the repair cost is given by $C = 50X + 10$ dollars, calculate $E(C)$.

Solution :

1. **Determine the value of the constant c :**
   The sum of probabilities must equal 1 : $\sum P(X = x_i) = 1$.

$$0.4 + 0.3 + c + 0.1 + 0.05 = 1 \text{(0.25 mark)}$$

$$0.85 + c = 1 \implies \mathbf{c = 0.15} \text{(0.25 mark)}$$

2. **Expression of the Cumulative Distribution Function (CDF), $F(x)$:**
   The CDF is defined as $F(x) = P(X \leq x)$:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.4 & \text{if } 0 \leq x < 1 \\ 0.7 & \text{if } 1 \leq x < 2 \\ 0.85 & \text{if } 2 \leq x < 3 \\ 0.95 & \text{if } 3 \leq x < 4 \\ 1 & \text{if } x \geq 4 \end{cases} \text{(1.5 marks)}$$

3. **Probability that at least 2 cracks are detected :**
   $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$P(X \geq 2) = 0.15 + 0.1 + 0.05 = \mathbf{0.30} \text{(0.5 mark)}$$

4. **Expected number of cracks $E(X)$ and Variance $V(X)$:**
   Expectation :

$$E(X) = \sum x_i P(x_i) = (0 \times 0.4) + (1 \times 0.3) + (2 \times 0.15) + (3 \times 0.1) + (4 \times 0.05)$$

$$E(X) = 0 + 0.3 + 0.3 + 0.3 + 0.2 = \mathbf{1.1} \text{(1 mark)}$$

   Variance :

$$E(X^2) = (0^2 \times 0.4) + (1^2 \times 0.3) + (2^2 \times 0.15) + (3^2 \times 0.1) + (4^2 \times 0.05) = 2.6$$

$$V(X) = E(X^2) - [E(X)]^2 = 2.6 - (1.1)^2 = 2.6 - 1.21 = \mathbf{1.39} \text{(1 mark)}$$

5. **Expected repair cost $E(C)$:**
   Given $C = 50X + 10$:

$$E(C) = E(50X + 10) = 50E(X) + 10 \text{(0.25 mark)}$$

$$E(C) = 50(1.1) + 10 = 55 + 10 = \mathbf{65} \text{ dollars(0.25 mark)}$$

Exercise n°3 (3 marks)
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Express the percentage of samples :

a) Having high clay content or low load-bearing capacity.

b) Not having high clay content.

c) Having neither high clay content nor low load-bearing capacity.

d) Having low load-bearing capacity but not high clay content

Exercise n°3 : (03 marks)

Solution :
Given : $P(H) = 0.40$, $P(L) = 0.18$, and $P(H \cap L) = 0.10$.

a) High clay or low load-bearing capacity $(H \cup L)$:
$P(H \cup L) = P(H) + P(L) - P(H \cap L)$

$$P(H \cup L) = 0.40 + 0.18 - 0.10 = 0.48 \implies \mathbf{48\%}$$

(0.75 mark)

b) Not having high clay content $(\bar{H})$:
$P(\bar{H}) = 1 - P(H)$

$$P(\bar{H}) = 1 - 0.40 = 0.60 \implies \mathbf{60\%}$$

(0.75 mark)

c) Neither high clay nor low capacity $(\overline{H \cup L})$:
$P(\overline{H \cup L}) = 1 - P(H \cup L)$

$$P(\overline{H \cup L}) = 1 - 0.48 = 0.52 \implies \mathbf{52\%}$$

(0.75 mark)

d) Low capacity but not high clay ($L \cap \bar{H}$) :
$P(L \cap \bar{H}) = P(L) - P(H \cap L)$

$$P(L \cap \bar{H}) = 0.18 - 0.10 = 0.08 \implies \mathbf{8}\%$$

(0.75 mark)

Module manager. Merini.A
Good luck