Mohamed Boudiaf University - M'sila
Faculty of Technology
Department of Civil Engineering-Department of Electrical Engineering

Module: Probability-Statistics

Chapter 6: Random variables

Merini Abdelaziz*

December 11, 2025

Contents

1 Introduction **2**

2 Definitions and properties **2**

3 Discrete Random Variable **3**
3.1 Discrete Probability Distributions 3
3.2 Graphical representation of the probability distribution 4
3.3 Cumulative distribution function for Discrete Random Variables 5
3.3.1 Properties 5
3.4 Relationship between distribution function and probability of X . 6
3.5 Mathematical Expectation 7
3.5.1 Interpretation 7
3.5.2 Properties of mathematical expectation 7
3.6 Variance and standard deviation of a discrete random variable . 7
3.6.1 Properties of V (X) 8

4 Continuous random variable **8**
4.1 Probability Density Function (PDF) 8
4.2 Cumulative Distribution Function (CDF) 9
4.3 Probability of an Interval 9
4.4 Moments of a Continuous Random Variable 11

*

4.4.1 Mathematical Expectation (Mean) 11
4.4.2 Variance 11
4.5 Moments 12
4.6 Notes 12
4.7 Covariance 12
4.8 4. Properties 12

1 Introduction

The notion of **random variable** plays an important role in probability. The majority of experiments can be modeled simply using a random variable.

2 Definitions and properties

Definition 1. *A **random variable X**, defined on a sample space Ω, is a function from Ω to the set of real numbers $\mathbb{R}$, where every possible outcome of the experiment (i.e., every element of Ω) is associated with a real number*

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \\ \omega &\longmapsto X(\omega) \end{aligned}$$

Remarks

• We use a capital letter, say X, to denote a random variable, and use the corresponding lower-case letter x to denote the realization (values) of the random variable.
• In general, the set of values taken by the random variable X is denoted by $X(\Omega)$ and is called the **range** of the random variable X.

• Let X and Y be two random variables defined on a sample space (Ω), then $X + Y$, XY, $X + \lambda$, and $\lambda.X$ (where λ is any real number) $upper(X;Y)$ and $lower(X;Y)$ are also random variables on Ω.
• If $X \neq 0$, then $\dfrac{1}{X}$ is random variable on Ω.

Example 1. *We toss a coin 2 times and observe the sequence of heads/tails. The sample space here may be defined as*
$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$.
Let X represent the number of heads that can come up.
$X(T,T)) = 0; X(H,T) = 1; X(T,H) = 1, X(H,H)) = 2.$
So the values that the random variable X can take are:0; 1; 2. i.e.

$X(\Omega) = \{0, 1, 2\}$

Example 2. *A box contains three balls numbered 2, 3 and 5. We draw successively with return* $\Omega = \{(2;2);(2;3);(2;5);(3;2);(3;3);(3;5);(5;2);(5;3);(5;5)\}$ *Let* Y *be the random variable indicating the sum of the points obtained*

$$Y : \Omega \longrightarrow \mathbb{R}$$
$$(i,j) \mapsto Y(i,j) = i + j$$

Therefore, $Y(\Omega) = \{4;5;6;7;8;10\}$.

Remark 1. *There are two types of random variables:*
1. Discrete random variables ;
2. Continuous random variables.

3 Discrete Random Variable

Definition 2. *A random variable* X *is said to be discrete random variable if its* ***range*** $X(\Omega)$ *includes finite number of values or countable infinite number of values.*

Example 3. *The random variables defined in the previous examples are discrete.*
X *can only take three values,* 0,*1 or* 2.
Y *can only take six values,* $4;5;6;7;8$ *or* 10.

3.1 Discrete Probability Distributions

The probability distribution of X is usually given in the form of the following table following table

X	x_1	x_2	$\dots$	x_n
P	p_1	p_2	$\dots$	p_n

The probability distribution satisfies the conditions :

1. $0 \leq P(X = x_i) \leq 1$.
2. $\sum_{i=1}^{n} P(X = x_i) = 1$.

Example 4. *In example 1, the probability distribution of boldsymbolX is*

X	*0*	*1*	*2*
P	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Example 5. *Taking example 2 again, the probability distribution of* Y *is*

Y	*4*	*5*	*6*	*7*	*8*	*10*
P	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

3.2 Graphical representation of the probability distribution

This is done using a bar chart with the values taken by the random variable on the abscissa and the corresponding probability values on the ordinate.

Example 6. *In the example of the coin toss: (example 1)*

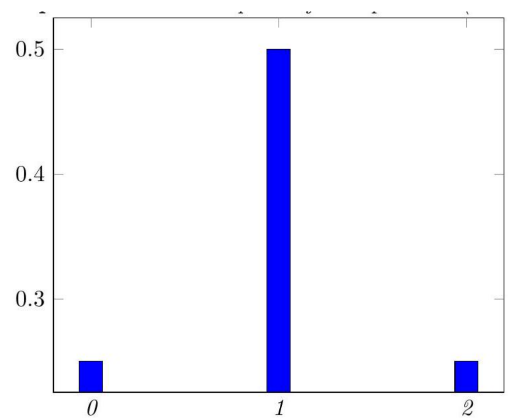


Figure 1: Distribution of X

Example 7. *Let's take the random variable Y indicating the sum of points obtained after two draws with a discount (example 2).*

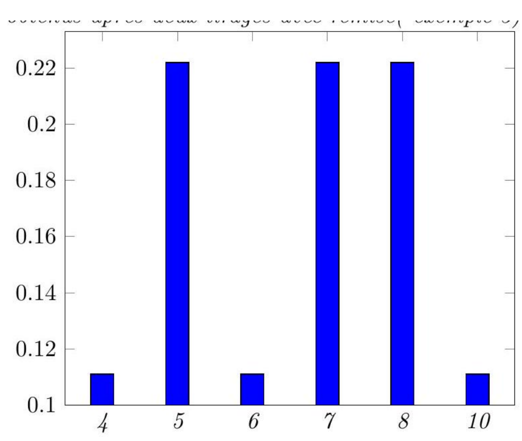


Figure 2: Distribution of Y

3.3 Cumulative distribution function for Discrete Random Variables

Definition 3. Let X be a discrete random variable on Ω. The cumulative distribution function of X is the function F_X defined by :

$$\begin{aligned} F_X &: \mathbb{R} \to [0;1] \\ & x \longmapsto F_X(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i) \end{aligned}$$

If X takes on only a finite number of values $x_1, x_1, ..., x_n$, then the distribution function is given by

$$F_X = \begin{cases} 0, & x < x_1 \\ P(X = x_1), & x_1 \leq x < x_2 \\ P(X = x_1) + P(X = x_2), & x_2 \leq x < x_3 \\ \vdots \\ P(X = x_1) + ... + P(X = x_n), & x_n \leq x \end{cases}$$

3.3.1 Properties

The distribution function F(x) has the following properties:

1) $F(x)$ is nondecreasing [i.e., $F(x) \leq F(y)$ if $x \leq y$]

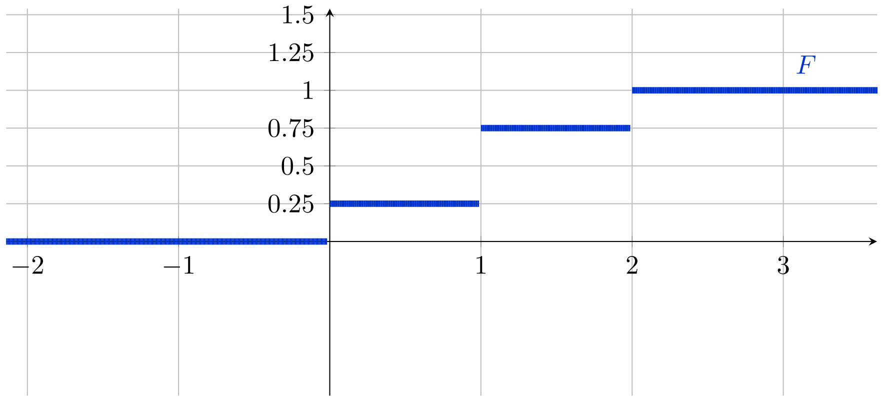
2) $\lim\limits_{x \to -\infty} F(x) = 0, \quad \lim\limits_{x \to +\infty} F(x) = 1.$

3) $F(x)$ is continuous from the right.

Example 8. *(a) Find the distribution function for the random variable X of Example 1.*
(b) Obtain its graph.
Answer
(a) The distribution function is

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x. \end{cases}$$

(b) The graph of F(x) is shown in Figure



Exercice 1. *(a) Find the distribution function for the random variable Y of Example 2.*
(b) Obtain its graph.
Answer *See worksheet.*

3.4 Relationship between distribution function and probability of X

Theorem 1. *Let $a; b \in R$ with $a < b$, X be a discrete random variable and F_X its function then :*

1) $P(a < X \leq b) = F_X(b) - F_X(a)$.

2) $P(a \leq X \leq b) = F_X(b) - F_X(a) + P(X = a)$.

3) $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$.

4) $P(a \leq X < b) = F_X(b) - F_X(a) + P(X = a) - P(X = b)$.

5) $P(a < X) = 1 - F_X(a)$.

6) $P(a \leq X) = 1 - F_X(a) + P(X = a)$.

Proof. Clearly it is sufficient to show the first equality, let $a; b \in R$ with a < b, we have :

$\{X \leq b\} = \{a < X \leq b\} \cup \{X \leq a\}$ and $\phi = \{a < X \leq b\} \cap \{X \leq a\}$
according to Axiom 3 (Chapter 4, page 6)

$P(X \leq b) = P(\{a < X \leq b\} \cup \{X \leq a\}) = P(\{a < X \leq b\}) + P(\{X \leq a\})$
$F_X(b) = P(a < X \leq b) - F_X(a)$
Therefore
$P(a < X \leq b) = F_X(b) - F_X(a)$ □

Example 9. *In example 2, calculate the following probabilities:*
$P(Y \leq 8) = F(8) = 8/9$.
$P(Y > 5) = 1 - P(Y \leq 5) = 1 - F(5) = 1 - 3/9 = 6/9$
$P(6 \leq Y \leq 7.5) = P(6 \leq Y \leq 7) = F(7) - F(6) + P(6) = 6/9 - 4/9 + 1/9 = 3/9$.

3.5 Mathematical Expectation

Definition 3. *For a discrete random variable X having the possible values* $x_1, , x_n$*, with respective probabilities* $P(X = x_1); P(X = x_n)$*: The* ***mathematical expectation*** *of X is :*

$$m = E(X) = \Sigma_{i=1}^{n} x_i P(X = x_i)$$

Example 10. *In example 1, we have*
$m = E(X) = \Sigma_{i=1}^{n} x_i P(X = x_i) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} = 1$

Example 11. *In example 2, we have*
$m = E(X) = \Sigma_{i=1}^{n} x_i P(X = x_i) = 7$

3.5.1 Interpretation

The expectation E[X] represents the mean value taken by X.

3.5.2 Properties of mathematical expectation

Let X and Y be two random variables defined on Ω, a and a two real constants.
1.$E(\alpha X) = \alpha E(X)$
2. $E(E(X)) = E(X)$; since E(X) is not a random variable.
3. $E(aX + bY) = aE(X) + bE(Y)$

3.6 Variance and standard deviation of a discrete random variable

Definition 4. *The variance is the quantity:*

$$V(x) = \Sigma_{i=1}^{n} P(X = x_i)(x_i - m)^2 \tag{1}$$

Definition 5. *The standard deviation is the quantity*

$$\sigma_x = \sqrt{V(x)} \tag{2}$$

Proposition 1. The variance of X, can be calculated as m = E(X) :

$$V(x) = \Sigma_{i=1}^{n} P(X = x_i) x_i - m^2 \tag{3}$$

Proof. exercice □

3.6.1 Properties of V (X)

Let X be a random variable defined on Ω, with α and β two real constants.
1)The variance of a constant is zero: $V(\alpha) = 0$
2)$V(X + \alpha) = V(X)$.
3)$V(\beta X + \alpha) = \beta^2 V(X)$

4 Continuous random variable

Definition 6. *A random variable X is said to be* ***continuous*** *when it can take on an infinite, non-countable number of values. This means that such a variable can take all the values of a real interval.*

Example 12. *Examples of continuous random variables include:*

- *The weight of a child at birth (e.g., between 2.7kg and 5.6kg).*
- *The time interval between two train passes.*
- *The lifetime in seconds of a mechanical part.*

4.1 Probability Density Function (PDF)

The function $f(x)$ is a **probability density function (PDF)** for the continuous random variable X, defined over the set of real numbers, if:

1. $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \displaystyle\int_{a}^{b} f(x)\,dx$

The first two conditions in the definition state the properties necessary for a function to be a valid pdf for a continuous random variable.
The third condition tells us how to use a pdf to calculate probabilities for continuous random variables.

Remark 2. *If X is a continuous random variable, then:*

$$\begin{aligned} P(a < X < b) &= P(a \leq X < b) \\ &= P(a < X \leq b) \\ &= P(a \leq X \leq b) \end{aligned}$$

This equality holds because the probability of X taking any exact value is zero in the continuous case. This is **not** the case for discrete random variables.

4.2 Cumulative Distribution Function (CDF)

The **cumulative distribution function (CDF)** of X is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

Properties of CDF

- $F(x)$ is non-decreasing.
- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to+\infty} F(x) = 1$
- If $F(x)$ is differentiable, then:

$$f(x) = \frac{d}{dx}F(x)$$

4.3 Probability of an Interval

For a continuous random variable X, the probability that X belongs to an interval $[x_1, x_2]$ can be obtained either by integrating the PDF or by using the CDF:

$$P(X \in [x_1, x_2]) = \int_{x_1}^{x_2} f(t)\,dt = F(x_2) - F(x_1)$$

Example 13. *Consider continuous random variable X, with PDF:*

$$f(x) = e^{-x}, \quad x \geq 0$$

The CDF is:

$$F(x) = 1 - e^{-x}, \quad x \geq 0$$

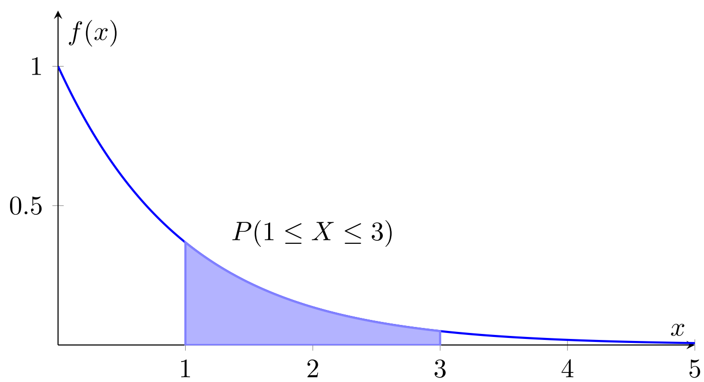
Probability on Interval $[1, 3]$

$$P(1 \leq X \leq 3) = \int_{1}^{3} e^{-t}\,dt = e^{-1} - e^{-3}$$

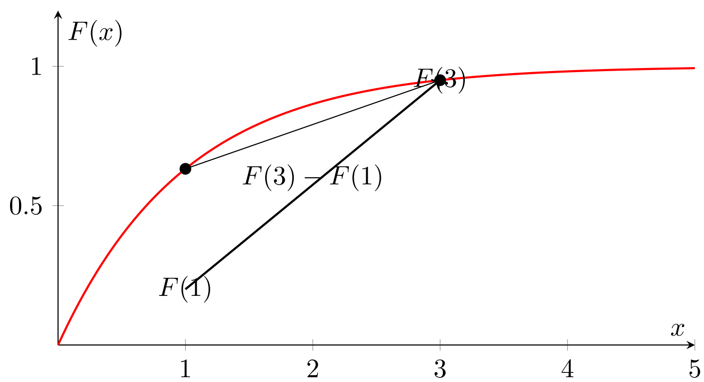
$$= F(3) - F(1) = (1 - e^{-3}) - (1 - e^{-1}) = e^{-1} - e^{-3}$$

Graphical Illustration

PDF of $Exp(1)$ with shaded probability



CDF of $Exp(1)$ showing $F(3) - F(1)$



Example 14. *Consider continuous random variable X, with PDF:*

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \textit{otherwise} \end{cases}$$

The CDF is:

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

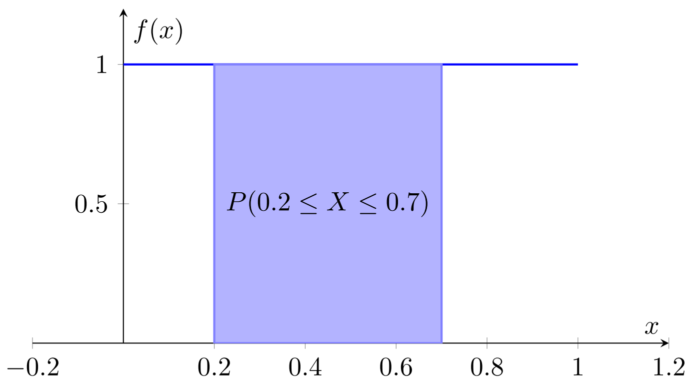
Probability on Interval $[0.2, 0.7]$

$$P(0.2 \leq X \leq 0.7) = \int_{0.2}^{0.7} 1\,dx = 0.5$$

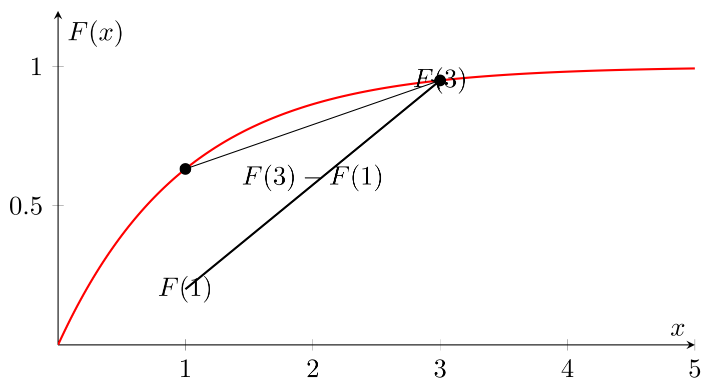
$$= F(0.7) - F(0.2) = 0.7 - 0.2 = 0.5$$

Graphical Illustration

PDF of $U(0,1)$ with shaded probability



CDF of $Exp(1)$ showing $F(3) - F(1)$



4.4 Moments of a Continuous Random Variable

4.4.1 Mathematical Expectation (Mean)

The **expected value** or **mean** of a continuous random variable X with probability density function $f(x)$ is defined as:

$$E[X] = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

4.4.2 Variance

The **variance** of X measures the dispersion around the mean:

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

Equivalently:

$$Var(X) = E[X^2] - (E[X])^2$$

4.5 Moments

Non-centered moments: The n-th non-centered moment of X is:

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

Centered moments: The n-th centered moment of X (around the mean μ) is:

$$\mu_n = E[(X-\mu)^n] = \int_{-\infty}^{\infty} (x-\mu)^n f(x)\, dx$$

4.6 Notes

- The first centered moment $\mu_1 = 0$.
- The second centered moment $\mu_2 = Var(X)$.
- Higher-order centered moments describe skewness (μ_3) and kurtosis (μ_4).

4.7 Covariance

For two continuous random variables X and Y with joint density $f_{X,Y}(x,y)$, the covariance is defined as:

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

Equivalently:

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

Where:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

4.8 4. Properties

- $Cov(X,X) = Var(X)$
- $Cov(X,Y) = Cov(Y,X)$
- If X and Y are independent, then $Cov(X,Y) = 0$