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Faculty of Technology
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Module: Probability-Statistics

Chapter 5: Conditional probability

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November 28, 2025

Contents

1 Introduction

Probability theory provides tools to quantify uncertainty in random events. In many situations, we do not have complete information about the outcome, but rather partial knowledge that can influence our expectations. Conditional probability offers a systematic way to incorporate this partial information into the calculation, allowing us to refine predictions and make more informed decisions.

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Example 1. *Suppose a medical test is positive. The probability that the patient actually has the disease depends not only on the test's accuracy but also on the prevalence of the disease in the population. This is a typical case where conditional probability is essential.*

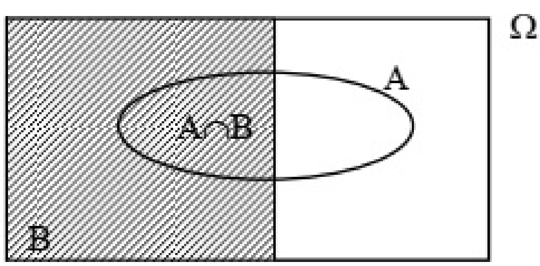
2 Conditional Probability

Definition 1. *Let A and B be two events in Ω such that $P(B) > 0$. The* ***conditional probability*** *of event A, given that event B has occurred is*

$$P_B(A) = \frac{P(A \cap B)}{P(B)}$$

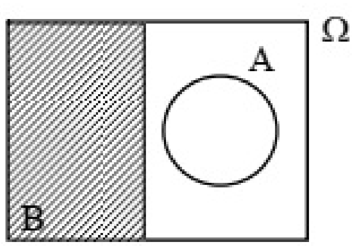
2.1 Remarks

1) $P_B(A))$ can be interpreted as the fact that Ω is restricted to B and the results of A are restricted to $A \cap B$.



2) If $A \cap B = \phi$ (A and B are mutually exclusive), A cannot occur if B has already occurred, and therefore

$$P_B(A))) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0.$$



3) In general $P_A(B)) \neq P_B(A)))$.

4) $P_B(\Omega) = \dfrac{P(B \cap \Omega)}{P(B)} = 1$.

Example 2. *Let's throw a dice and consider the events* $A = \{5\}$ *and* $B = \{1, 3, 5\}$*:*

$$P(A) = \frac{1}{6} \quad , P(B) = \frac{3}{6} \quad \textit{et } P(A \cap B) = \frac{1}{6}$$

- *The probability of '5 coming out' knowing that it is 'an odd number' is :*

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

- *The probability of an 'even number' coming out knowing that it is '5' is*

$$P_A(B) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}} = 1$$

Note that $P_B(A)) \neq P_A(B)$.

- *The probability that '5 does not come out' knowing that it is 'an odd number' is*

$$P_B(\bar{A}) = 1 - P_B(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

2.2 Conditional probabilities are probabilities

Conditional probability satisfies all the properties of probability.
For fixed B with $P(B) > 0$, the function $P_B(\cdot)$ satisfies the axioms of probability:

- $P_B(A) \geq 0$.
- $P_B(\Omega) = 1$.
- If A_1, A_2 are disjoint, then $P_B(A_1 \cup A_1) = P_B(A_1) + P_B(A_2)$.

Properties

1. Let A and B be two events of Ω such that $P(A) \neq 0$ and $P(B) \neq 0$. We have:

$$P(A \cap B) = P_A(B) \times P(A) = P_B(A) \times P(B)$$

2. Let A, B, and C be three events of Ω with $P(A) \neq 0$.

- if $C \sqsubseteq B$, then $P_A(C) \leq P_A(B))$
- $P_A((B \cup C)) = P_A(B) + P_A(C) - P_A((B \cap C))$.
- $P_A(B) + P_A(\bar{B}) = 1$.

3. $P_A(A) = \frac{P(A \cap A)}{P(A)} = 1$.

2.3 Theorems on Conditional Probability

Definition 2. *Let n be an integer equal to or greater than 2. To say that n events $B_1, B_2, \ldots, B_n$ form a partition of the Ω universe means that:*

- *For all $i \in \{1; 2; \ldots; n\}, B_i \neq \emptyset$.*
- $\forall i \neq j : B_i \cap B_j = \emptyset$..
- $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$.

2.3.1 Law of total probability

Theorem 1. ***(Law of total probability)***
Let n events $B_1, B_2, \ldots, B_n$ have non-zero probabilities and form a partition of Ω.
For all event A of Ω we have

$$\begin{aligned} P(A) &= P\left(A \cap B_1\right) + P\left(A \cap B_2\right) + \ldots + P\left(A \cap B_n\right) \\ &= P\left(B_1\right) P_{B_1}(A) + P\left(B_2\right) P_{B_2}(A) + \ldots + P\left(B_n\right) P_{B_n}(A) \end{aligned}$$

Remark . *Since ($B \cup \bar{B} = \Omega$ and $B \cup \bar{B} = \phi$), we can apply the total probability formula:*

$$P(A) = P(B) \times P_B(A) + P(\bar{B}) \times P_{\bar{B}}(A)$$

2.3.2 Formula of compound probabilities

Theorem 2. Let A, B, C, D be events in a universe Ω.
a) $P(A \cap B) = P(A)P_A(B)$.
b) $P(A \cap B \cap C) = P(A)P_A(B)P_{(B \cap A)}(C)$
c) $P(A \cap B \cap C \cap D) = P(A)P_A(B)P_{(A \cap B)}(C)P_{(A \cap B \cap C)}(D)$.

Proof. a) According to the definition

$$P_A(B) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)P_A(B)$$

b) According to the definition

$$\begin{aligned} P_{(A \cap B)}(C) &= \frac{P(A \cap B) \cap C)}{P(A \cap B)} \\ &\Longrightarrow P(A \cap B \cap C) = P(A \cap B)P_{(A \cap B)}(C) = P(A)P_A(B)P_{(A \cap B)}(C) \end{aligned}$$

□

Example 3. . *Consider an urn containing 4 white balls and 6 red balls. 3 balls are drawn one after the other. What is the probability of obtaining a draw consisting of 3 white balls?*

Answer Let W_i be the event: the i-th ball drawn is white. Clearly

$$P\left(W_1\right) = \frac{4}{10}$$

$$P_{W_1}\left(W_2\right) = \frac{3}{9}$$

$$P_{\left(W_2 \cap W_1\right)}\left(W_3\right) = \frac{2}{8}.$$

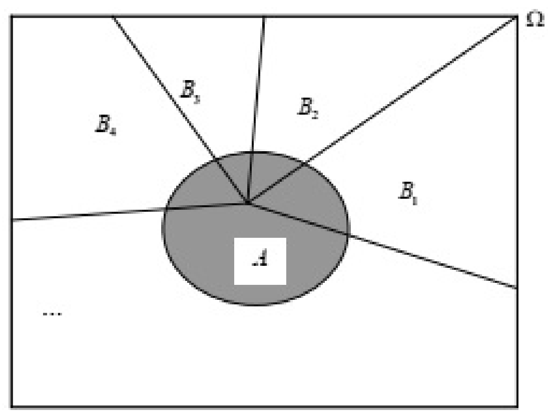
By compound probabilities, we obtain

$$P\left(W_1 \cap W_2 \cap W_3\right) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$$

3 Bayes' Theorem

Theorem 2. ***(Bayes' Theorem)*** *Let $B_1, B_2, \ldots, B_n$, n events disjoint two by two i.e. $\forall i \neq j : Bi \cap Bj = \emptyset$ and such that $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$.*
Let A be an event
Then

$$\begin{aligned} P_A\left(B_i\right) &= \frac{P\left(A \cap B_i\right)}{P(A)} \\ &= \frac{P\left(B_i\right) P_{B_i}(A)}{\sum_{j=0}^{n} P\left(B_j\right) P_{B_j}(A)} \end{aligned}$$



Proof. We have $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$ and $A \cap \Omega = A$.
So

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = A \Leftrightarrow$$
$$(A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) = A.$$

So,

$$\begin{aligned} P\left((A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)\right) = P(A) \Leftrightarrow \\ P\left(A \cap B_1\right) + P\left(A \cap B_2\right) + \ldots + P\left(A \cap B_n\right) = P(A) \Leftrightarrow \\ P\left(B_1\right) \times P_{B_1}\left(A\right) + \ldots + P\left(B_n\right) \times P_{B_n}\left(A\right) = P(A) \Leftrightarrow \\ \sum_{k=0}^{n} P\left(B_k\right) P_{B_k}\left(A\right) = P(A). \end{aligned}$$

So..,

$$P\left(B_i / A\right) = \frac{P\left(B_i\right) P_{B_i}\left(A\right)}{\sum_{j=0}^{n} P\left(B_j\right) P_{B_j}\left(A\right)}$$

□

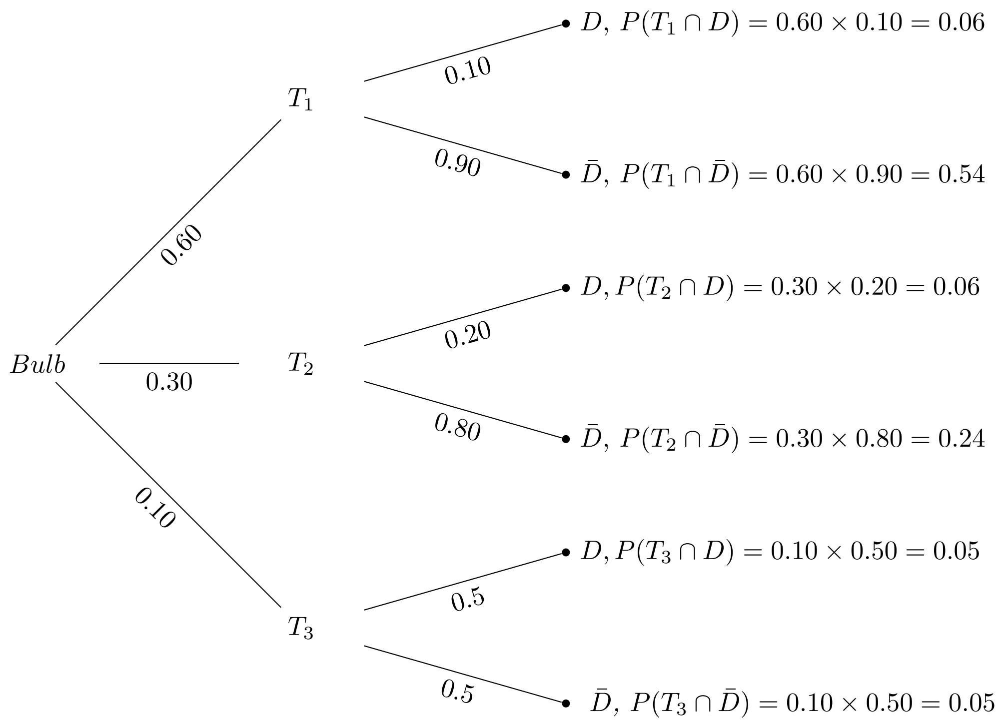
Remark 1. *Bayes' theorem can be used to calculate* $P_A\left(B_j\right)$ *knowing* $P_{B_j}\left(A\right), \forall j \in \{1; 2; \ldots; n\}$.

Example 4. . *A company uses three types of bulb* T_1, T_2 *and* T_3*, in proportions of* $60\%, 30\%$ *and* 10% *respectively. The probability of these bulbs working is respectively* $90\%, 80\%$ *et* 50%.

What is the probability of a faulty bulb coming from T_1*?*

Réponse *:*
$T_1 = \ll$ *use a bulb* $T_1 \gg$ *so* $P\left(T_1\right) = 0.60$
$T_2 = \ll$ *use a bulb* $T_2 \gg$ *so* $P\left(T_2\right) = 0.30$
$T_3 = \ll$ *use a bulb* $T_3 \gg$ *so* $P\left(T_3\right) = 0.10$
$D =$ *defective bulb.* $\bar{D} =$ *non-defective bulb.*

We're looking for $P_D(T_1) =$?
We have



then

$$\begin{aligned} P(D) &= P(T_1) P_{T_1}(D) + P(T_2) P_{T_2}(D) + P(T_3) P_{T_3}(D) \\ &= 0.60 \times 0.10 + 0.30 \times 0.20 + 0.10 \times 0.50 \\ &= 0.17 \end{aligned}$$

so,

$$P_D(T_1) = \frac{p(T_1 \cap D)}{P(D)} = \frac{0.60 \times 0.10}{0.17} = \frac{6}{17} \simeq 0,353.$$

4 Independence of events

Definition 3. *(Independence of two events). Events A and B are independent if*

$$P(A \cap B) = P(A)P(B)$$

If $P(A) > 0$ and $P(B) > 0$, then this is equivalent to $P_B(A) = P(A)$; and also equivalent to $P(B/A) = P(B)$:

In words, two events are independent if we can obtain the probability of theirintersection by multiplying their individual probabilities. Alternatively, A and B are independent if learning that B occurred gives us no information that would change our probabilities for A occurring (and vice versa).

Remark 2. *Independence is completely different from disjointness.*
If A and B are disjoint, then $P(A \cap B) = 0$*, so disjoint events can be independent only if P (A) = 0 or P (B) = 0.*
Knowing that A occurs tells us that B definitely did not occur, so A clearly conveys information about B, meaning the two events are not independent (except if A or B already has zero probability).

Example 5. *Two dice are thrown one after the other. The following events are noted :*
$A = \ll$ *the sum of the dice is* $7 \gg$*, then,* $A = \{(1;6),(2;5),(3;4),(6;1),(5;2),(4;3)\}$
ainsi $P(A) = \frac{6}{36} = \frac{1}{6}$
$B =$ *«the first die gives* $4 \gg$*, which gives* $B = \{(4;6),(4;5),(4;4),(4;1),(4;2),(4;3)\}$
so $P(B) = \frac{6}{36} = \frac{1}{6}$
$A \cap B = \ll$ *the sum of the dice is 7 and the first die gives* 4*, so* $A \cap B = \{(4;3)\}$
as a result $P(A \cap B) = \frac{1}{36}$
In this example, we can see that

$$P(A \cap B) = P(A) \times P(B)$$

so events A and B are independent.

Proposition 1. *If A and B are independent, then A and* $\bar{B}$ *are independent,* $\bar{A}$ *and B are independent, and* $\bar{A}$ *and* $\bar{B}$ *are independent.*

Proof. exercice. □