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Module : Probability-Statistics

Chapter 3 : COMBINATORIAL ANALYSIS

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1 Introduction

Combinatorial analysis, the mathematical study of counting, plays a crucial role in determining the number of possible outcomes in any experiment. Mastery of these enumeration techniques is fundamental for calculating probabilities accurately and efficiently.

1.1 Definitions :

• A set Ω is finite when it has a finite number of elements.

The number of elements of Ω is called the cardinal of the set and it is noted :

$\sharp\Omega$ or $|\Omega|$.

• To count is to determine the number of elements in a finite set, i.e. to determine its cardinal.

Example 1. *If* $\Omega = \{heads, tails\}$*, its cardinal is* $|(\Omega)| = 2.$.

- If Ω*= {1, 2, 3, 4, 5, 6},* $|(\Omega)| = 6$.

If Ω *is the set of integers,* $|(\Omega)| = +\infty$*.* $card(\phi) = 0$.

The set Omega of players in a football team is a finite set. . Then $|(\Omega)| = 11$.

Definition 1. *We say that two sets* Ω_1 *and* Ω_2 *are* ***disjoint****, if they have no elements in common, i.e.*

$$\Omega_1 \cap \Omega_2 = \phi$$

1.2 Factor notation

Definition 2. *Let* n $n \in \mathbb{N}$*. The product of the integers from 1 to* n *is called* n *factorial, denoted* n*!*

$$n! = \begin{cases} 1 \quad \textit{if } n = 0 \\ n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1 \quad \textit{if } n > 0 \end{cases}$$

Example 2. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. $6! = 6 \times 5! = 720$.

$0! = 1$.

Remark 1. $\dfrac{n!}{(n-1)!} = n, \qquad n! = n \times (n-1)!$

2 Basic Counting Principles

The basic counting principles are **the product rule** and **the sum rule** will be presented and show how can be used tosolve many different counng problems.

2.1 Sum Rule

Suppose that a set Ω is partitioned into pairwise disjoint parts Ω_1, Ω_2,...,Ω_k , .

The sum Rule claims that in this case

$$|\Omega| = |\,(\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k)\,| = |\,(\Omega_1)\,| + |\,(\Omega_2)\,| + ... + |\,(\Omega_k)\,|$$

Example 3. *A vending machine offers 5 types of chips and 3 types of chocolate bars. If you can choose either a bag of chips or a chocolate bar, how many snack options do you have ?*

Solution :

Chips : 5 options Chocolate bars : 3 options Total options : 5+3=8

Example 4. *Let* $\Omega_1 = \{a, b, c, d\}$ *and* $\Omega_2 = \{g, h, k\}$. *we have* $\Omega_1 \cap \Omega_2 = \phi$ *, then*

$$|\,(\Omega_1 \cup \Omega_2)\,| = |\,(\Omega_1)\,| + |\,(\Omega_2)\,| = 4 + 3 = 7.$$

Example 5. *Let* Ω *be the set of students attending the combinatorics lecture. It can be partitioned into parts* Ω_1 *and* Ω_2 *where :*

$\Omega_1 =$ *set of students that like easy examples.*

$\Omega_2 =$ *set of students that don't like easy examples.*

If $|\Omega_1| = 22$ *and* $|\Omega_2| = 8$ *then we can conclude* $|\Omega| = 8 + 22 = 30$.

2.2 Product Rule

If k experiments that are to be performed are such that :

the first one may result in any of n_1 possible outcomes ;

and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment ;

and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment ;

and if ...,

then there is a total of $n_1 \times n_2 \times ... \times n_k$ possible outcomes of the k experiments.

Example 6. *We want to print a car number plate with, from left to right, 2 distinct letters and 3 digits, the first of which is not zero (a global experiment). How many plates of this type are there ?* $(CH124, DE665, ..., ...)$

Solution 1.

letter	*letter*	*number*	*number*	*number*
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
26	*25*	*9*	*10*	*10*

According to the Multiplication Principle, the possible number of plates of this type is

$$26 \times 25 \times 9 \times 10 \times 10 = 585000$$

Example 7. *Three 6-sided dice are rolled in succession (a global experiment).*
How many possible outcomes are there ?

Solution 2. *For D_1, we have (6 distinct numbers)*
For D_2, we have (6 distinct numbers)
For D_3, we have (6 distinct numbers)
According to the Multiplication Principle, the number of possible results is

$$6 \times 6 \times 6 = 216.$$

Example 8. *You are making a sandwich. You have 2 types of bread (white, wheat) and 3 types of fillings (turkey, chicken, vegetables). How many different sandwiches can you make if you choose one type of bread and one type of filling ?*
Solution :
Bread options : 2
Filling options : 3
Total sandwiches : $3 \times 2 = 6$

3 Permutations

3.1 Permutations without Repetitions

Definition 3. *A permutation refers to the number of ways to arrange a set of objects in a specific order, where all objects are **distinct** and **order matters.***

Example 9. *The possible permutations of the 3 letters a, b, c are : abc, bca, cab, bac, acb, cba.*

Proposition 1. *The number of permutations of n objects taken all at a time :*

$$P_n = n!$$

Example 10. *In the example 9, on a $n = 3$, donc $P_3 = 3! = 6$.*

3.2 Permutations with repetitions

Definition 4. *If we classify in a particular order n elements of which*
n_1 are identical of type 1,
n_2 are identical of type 2,
.......,
n_k are identical of type k,
we form a permutation with repetitions of these $n = n_1 + n_2 + ... + n_k$ elements.

Proposition 2. *The number of permutations with repetitions is*

$$\bar{P}_n(n_1, n_2, ..., n_k) = \frac{n!}{n_1!n_2!...n_k!}$$

Example 11. *How many different words can you make with all the letters in the word « ERRER »*

We have 5 letters ($n = 5$) some of which are similar.

the letter 'E' appears twice so $n_1 = 2$, the letter 'R' appears three times so $n_2 = 3$, then

$$\bar{P}_5(2;3) = \frac{5!}{2!3!} = 10$$

RRREE, RRERE, RREER, RERRE, RERER, REERR, ERRRE, ERRER ; ERERR ; EERRR.

Example 12. *A chess tournament has 10 competitors, of which 4 are Russian, 3 are from theUnited States, 2 are from Great Britain, and 1 is from Brazil. If the tournamentresult lists just the nationalities of the players in the order in which they placed,how many outcomes are possible ?*

Solution *There are $\frac{10!}{2!3!1!4!} = 12600$ possible outcomes*

Remark 2. *a)* $\bar{P}_n\left(n_1, n_2, ..., n_k\right) < P_n$ *.*

b) The bar above the P stands for 'with repetition'.

3.3 Circular permutation

Definition 5. *Circular permutation is the total number of ways in which n distinct objects can be arranged around a fixed circle.*

Proposition 3. *The number of circular permutations of n objects is equal to*

$$(n-1)!$$

Example 13. *How many ways can you arrange 5 people :*
a) On a line ?
b) Around a round table ?

Solution 3. *a) 5 !=120.*
b)(5-1) !=4 !=24.

3.4 r-Permutation (arrangement)

Definition 6. *Let r be a positive integer* $(r \leq n)$*. By an r-permutation (arrangement) of a set* Ω *of n elements, we understand an ordered arrangement of r of the n elements.*

Example 14. *If* $\Omega = \{a, b, c\}$ *then the three 1-permutations of* Ω *are a , b, c,*
the six 2-permutations of S are ab ac ba be ca cb,
and the six 3-permutations of S are abc acb bac bca cab cba.

Proposition 4. *The number of r-permutations froma set containing n elements is given by*

$$\begin{aligned} A_r^n = P(n,r) &= \frac{n!}{(n-r)!} \\ &= n\,(n-1)\,...\,(n-r+1) \end{aligned}$$

Remark 3. *If* $r = n$*, we have* $A_n^n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!,$

Example 15. *With the letters of the word « RELATION », how many can you form, with or without meaning, different 3-letter words ?*

Answer : $n=8$ *et* $r=3$ *,there are* $A_5^8 = \dfrac{8!}{(8-3)!} = 8\times7\times6 =$ 336 *possible words*

Proposition 5. *The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is*

$$\bar{A}_r^n n^r$$

Example 16. *How many 3 digit numbers can be formed with the digits 1 and 2 ?*

Answer : $n=2$, *there are* $2^3=8$. *numbers 111, 112, 121, 211, 122, 222, 221,212.*

Example 17. *How many two-digit numbers can you make using the digits 5, 6, 7, 8, 9 ?*

Answer : $n=5$, $r=2$, $5^2=25$.

4 Combination

4.1 Simple Combination

Definition 7. *A combination of a set of elements is an arrangement where each element is used once, and order is not important.*

Example 18. *Four people* $\{1;2;3;4\}$ *want to play doubles table tennis. doubles. How many different teams can they form ?*

Solution : it's a simple combination : $\{1;2\},\{3;4\},\{1;3\},\{2;4\},\{1;4\},\{2:3\}$, *6 teams can be formed.*

Proposition 6. *The Number of Combinations of n Objects Taken r at a Time*

$$\begin{aligned}\left(\begin{array}{c} n \\ r \end{array}\right) &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{P(n,r)}{r!}\end{aligned}$$

Remark 4. *1)We also note* C_r^n *or* $\left(\begin{array}{c} n \\ r \end{array}\right)$.

2) $C_r^n \leq A_r^n$

Example 19. *A committee of 3 is to be formed from a group of 24 people. How many different committees are possible ?*

Solution

There are

$$C_3^{24} = \frac{24!}{3!\,(24-3)!} = \frac{24!}{3!21!} = \frac{24 \times 23 \times 22}{6} = 2024$$

possible committees

Properties

1) $C_0^n = C_n^n) = 1 \quad , C_r^n = C_r^{n-r})$ (symmetry formula)
2) $C_1^n = n.$
3) $C_{r+1}^{n+1} = C_r^n + C_{r+1}^n$ (Pascal's triangle).
4) $\sum\limits_{i=0}^{n} C_i^n = 2^n$

Démonstration. exercice. □

Example 20. $C_0^3 = \dfrac{3!}{0!\,(3-0)!} = 1$; $C_3^3 = 1$; $C_2^3 = C_1^3 = 3$

4.2 Combinations with Repetition

Definition 8. *A combination with repetition of r objects from n is a way of selecting r objects from a list of n. The selection rules are :*

1) the order of selection does not matter

2) each object can be selected more than once.

Example 21. *17 Let $\Omega = a, b, c$. We want to obtain the list of combinations of k=4 elements taken from the set E, i.e. generate the list of 4-combinations with repetition :*

(a, a, a, a), (a, a, a, b), (a, a, a, c), (a, a, b, b), (a, a, b, c), , (a, a, c, c) , (a, b, b, b) , (a, b, b, c) , (a, b, c, c) , (a, c, c, c) ,(b, b, b, b) , (b, b, b, c) , (b, b, c, c) , (b, c, c, c) , (c, c, c, c)

Proposition 7.

Number of combinations with repetition of r objects from n is

$$\begin{pmatrix} n+r-1 \\ r \end{pmatrix} = \frac{(n+r-1)!}{r!(n-1)!}$$

Example 22. *In the previous example , we have* $\begin{pmatrix} 3+4-1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15$

4.3 Pascal's Trianlge

The relationship $C_{r+1}^{n+1} = C_r^n + C_{r+1}^n$
allows practical determination of the various coefficients C_r^n using Pascal's triangle.

If $r > n$, , we put $C_r^n = 0$

$n\backslash r$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

4.4 Newton's Binomial Theorem

Let n be a positive integer. Then, for all x and y,

$$(a+b)^n = \sum_{i=0}^{n} C_n^i a^{n-i} b^i$$

Example 23. $(a+b)^2 = C_2^0 a^2 + C_2^1 ab + C_2^2 a^0 b^2 = 1a^2 + 2ab + 1b^2$
$(a+b)^3 = C_3^0 a^3 + C_3^1 a^2 b + C_3^2 a^1 b^2 + C_3^3 a^0 b^2 = 1a^3 + 3a^2 b + 3a^1 b^2 + 1b^3$.