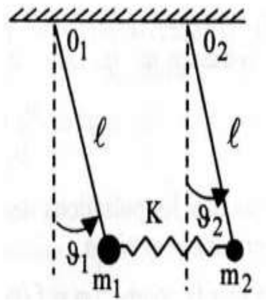
University of M'sila — university years 2024/2025
Faculty of technology — field: ST
Department of Hydraulics and Civil Engineering — module: waves and vibrations

Tutorials (TD) Series №5

Linear systems with two degrees of freedom

Exercise №1: (elastic coupling)

Two pendulums of the same length l but different masses $(m_1 > m_2)$ oscillate around their respective axes $O_1 et O_2$(low amplitudes) being coupled by a stiffness spring K arranged according to the figure below.



1. Establish the differential equations of movement of the system.
2. Make the following variable change: $\theta_1 = \beta_1 + \beta_2 et \theta_2 = \beta_1 - \frac{m_1}{m_2}\beta_2$
3. Then give the proper pulsations $\omega_1 et \omega_2$ of the system as well as the general solution of the movement knowing that at the initial moment $t = 0$ we had:

$\theta_1(0) = \theta_0$; $\dot{\theta}_1(0) = 0$; $\theta_2(0) = 0$; $\dot{\theta}_2(0) = 0$

Solution (Exercise №1):

1- the differential equations of movement of the system:

- kinetic energy: $\mathrm{T} = \mathrm{T}_{m1} + \mathrm{T}_{m2}$

$$\begin{cases} \mathrm{T}_{m1} = \frac{1}{2}m_1 v_1^2 \\ \mathrm{T}_{m2} = \frac{1}{2}m_2 v_2^2 \end{cases} \Rightarrow \begin{cases} \mathrm{T}_{m1} = \frac{1}{2}m_1 \ell^2 \dot{\theta}_1^2 \\ \mathrm{T}_{m2} = \frac{1}{2}m_2 \ell^2 \dot{\theta}_2^2 \end{cases} \Rightarrow T = \frac{1}{2}m_1 \ell^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \ell^2 \dot{\theta}_2^2$$

- potential energy: $\mathrm{U} = \mathrm{U}_{m1} + \mathrm{U}_{m2} + U_k$

$$\begin{cases} \mathrm{U}_{m1} = m_1 g\ell\,(1 - cos\theta_1) \\ \mathrm{U}_{m2} = m_2 g\ell\,(1 - cos\theta_2) \\ \mathrm{U}_k = \frac{1}{2}k\ell^2\,(sin\theta_1 - sin\theta_2)^2 \end{cases}$$

$$\theta_1 \ll;\ \theta_2 \ll \Rightarrow \begin{cases} cos\theta_1 \approx 1 - \frac{{\theta_1}^2}{2} \ ;sin\theta_1 \approx \theta_1 \\ cos\theta_2 \approx 1 - \frac{{\theta_2}^2}{2} \ ;sin\theta_2 \approx \theta_2 \end{cases} \Rightarrow \begin{cases} \mathrm{U}_{m1} = \frac{1}{2}m_1 g\ell{\theta_1}^2 \\ \mathrm{U}_{m2} = \frac{1}{2}m_2 g\ell{\theta_2}^2 \\ \mathrm{U}_k = \frac{1}{2}k\ell^2\,(\theta_1 - \theta_2)^2 = \frac{1}{2}k\ell^2{\theta_1}^2 + \frac{1}{2}k\ell^2{\theta_2}^2 - k\ell^2\theta_1\theta_2 \end{cases}$$

$$\Rightarrow U = \frac{1}{2}(m_1 g\ell + k\ell^2){\theta_1}^2 + \frac{1}{2}(m_2 g\ell + k\ell^2){\theta_2}^2 + k\ell^2\theta_1\theta_2$$

Lagrange function $\mathcal{L} = \mathrm{T} - \mathrm{U} \Rightarrow \mathcal{L} = \frac{1}{2}m_1\ell^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell^2\dot{\theta}_2^2 - \frac{1}{2}(m_1 g\ell + k\ell^2){\theta_1}^2 - \frac{1}{2}(m_2 g\ell + k\ell^2){\theta_2}^2 + k\ell^2\theta_1\theta_2$

- the Lagrange equations: $\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \end{cases}$

$$\Rightarrow \begin{cases} m_1\ell^2\ddot{\theta_1} + (m_1 g\ell + k\ell^2)\theta_1 - k\ell^2\theta_2 = 0 \\ m_2\ell^2\ddot{\theta_2} + (m_2 g\ell + k\ell^2)\theta_2 - k\ell^2\theta_1 = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\theta_1} + \dfrac{(m_1 g\ell + k\ell^2)}{m_1\ell^2}\,\theta_1 - \dfrac{k\ell^2}{m_1\ell^2}\theta_2 = 0 \\ m_2\ell^2\ddot{\theta_2} + \dfrac{(m_2 g\ell + k\ell^2)}{m_2\ell^2}\theta_2 - \dfrac{k\ell^2}{m_2\ell^2}\theta_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\theta_1} + \left(\dfrac{g}{\ell} + \dfrac{k}{m_1}\right)\,\theta_1 - \dfrac{k}{m_1}\theta_2 = 0 \\ \ddot{\theta_2} + \left(\dfrac{g}{\ell} + \dfrac{k}{m_2}\right)\theta_2 - \dfrac{k}{m_2}\theta_1 = 0 \end{cases}$$

2-Make the following variable change: $\begin{cases} \theta_1 = \beta_1 + \beta_2 \\ \theta_2 = \beta_1 - \frac{m_1}{m_2}\beta_2 \end{cases} \Rightarrow \begin{cases} \ddot{\theta_1} = \ddot{\beta_1} + \ddot{\beta_2} \\ \ddot{\theta_2} = \ddot{\beta_1} - \frac{m_1}{m_2}\ddot{\beta_2} \end{cases}$

$$\Rightarrow \begin{cases} (\ddot{\beta_1} + \ddot{\beta_2}) + \left(\dfrac{g}{\ell} + \dfrac{k}{m_1}\right)\,(\beta_1 + \beta_2) - \dfrac{k}{m_1}(\beta_1 - \dfrac{m_1}{m_2}\beta_2) = 0 \\ (\ddot{\beta_1} - \dfrac{m_1}{m_2}\ddot{\beta_2}) + \left(\dfrac{g}{\ell} + \dfrac{k}{m_2}\right)(\beta_1 - \dfrac{m_1}{m_2}\beta_2) - \dfrac{k}{m_2}(\beta_1 + \beta_2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left[\ddot{\beta_1} + (\dfrac{g}{\ell} + \dfrac{k}{m_1} - \dfrac{k}{m_1})\beta_1\right] + \left[\ddot{\beta_2} + (\dfrac{g}{\ell} + \dfrac{k}{m_1} + \dfrac{k}{m_2})\beta_2\right] = 0 \\ \left[\ddot{\beta_1} + (\dfrac{g}{\ell} + \dfrac{\ddot{k}}{m_2} - \dfrac{k}{m_2})\beta_1\right] - \dfrac{m_1}{m_2}\left[\ddot{\beta_2} + \left(\dfrac{g}{\ell} + \dfrac{k}{m_1} + \dfrac{k}{m_2}\right)\beta_2\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left[\ddot{\beta_1} + (\dfrac{g}{\ell})\beta_1\right] + \left[\ddot{\beta_2} + (\dfrac{g}{\ell} + \dfrac{k}{m_1} + \dfrac{k}{m_2})\beta_2\right] = 0 \\ \left[\ddot{\beta_1} + (\dfrac{\ddot{g}}{\ell})\beta_1\right] - \dfrac{m_1}{m_2}\left[\ddot{\beta_2} + \left(\dfrac{g}{\ell} + \dfrac{k}{m_1} + \dfrac{k}{m_2}\right)\beta_2\right] = 0 \end{cases}$$

So ;

$$\Rightarrow \begin{cases} \ddot{\beta_1} + (\dfrac{g}{\ell})\beta_1 = 0 \\ \ddot{\beta_2} + \left(\dfrac{g}{\ell} + \dfrac{k}{m_1} + \dfrac{k}{m_2}\right)\beta_2 = 0 \end{cases}$$

3- The propre pulsations:

$\omega_1 = \sqrt{\frac{g}{\ell}}$ and $\omega_2 = \sqrt{\frac{g}{\ell} + \frac{k}{m_1} + \frac{k}{m_2}}$

$$\Rightarrow \begin{cases} \ddot{\beta_1} + {\omega_1}^2\,\beta_1 = 0 \\ \ddot{\beta_2} + {\omega_2}^2\beta_2 = 0 \end{cases}$$

-the general solution of the motion:

$$\Rightarrow \begin{cases} \beta_1(t) = A_1 cos(\,\omega_1 t + \varphi_1) \\ \beta_2(t) = A_2 cos(\,\omega_2 t + \varphi_2) \end{cases}$$

So; $\begin{cases} \theta_1 = \beta_1 + \beta_2 \\ \theta_2 = \beta_1 - \frac{m_1}{m_2}\beta_2 \end{cases} \Rightarrow \begin{cases} \theta_1(\mathrm{t}) = A_1cos(\omega_1 t + \varphi_1) + A_2cos(\omega_2 t + \varphi_2) \\ \theta_2(\mathrm{t}) = A_1cos(\omega_1 t + \varphi_1) - \frac{m_1}{m_2}A_2cos(\omega_2 t + \varphi_2) \end{cases}$

the initial moment $t = 0$ we had: $\theta_1(0) = \theta_0$; $\dot{\theta_1}(0) = 0$; $\theta_2(0) = 0$; $\dot{\theta_2}(0) = 0$

$$\Rightarrow \begin{cases} \theta_1(0) = A_1cos(\varphi_1) + A_2cos(\varphi_2) \\ \theta_2(0) = A_1cos(\varphi_1) - \dfrac{m_1}{m_2}A_2cos(\varphi_2) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\theta_1}(\mathrm{t}) = -\omega_1 A_1 sin\big(\omega_1 t + \varphi_1\big) - \omega_2 A_2 sin(\omega_2 t + \varphi_2) \\ \dot{\theta_2}(\mathrm{t}) = -\omega_1 A_1 sin(\omega_1 t + \varphi_1) - \omega_2 \dfrac{m_1}{m_2} A_2 sin(\omega_2 t + \varphi_2) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\theta_1}(0) = -\omega_1 A_1 sin\big(\varphi_1\big) - \omega_2 A_2 sin(\varphi_2) \\ \dot{\theta_2}(\mathrm{t}) = -\omega_1 A_1 sin(\varphi_1) - \omega_2 \dfrac{m_1}{m_2} A_2 sin(\varphi_2) \end{cases}$$

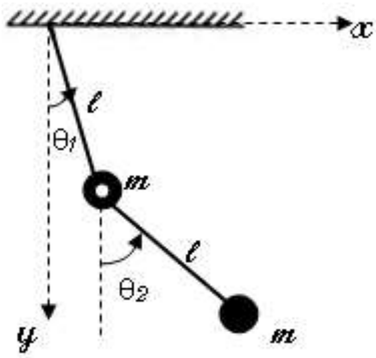
Find some: $\varphi_1 = \varphi_2 = 0$; $A_1 = \frac{\mu}{m_2}\theta_0$; $A_2 = \frac{\mu}{m_1}\theta_0$;

$\mu = (\dfrac{1}{m_1} + \dfrac{1}{m_2})$

$$\Rightarrow \begin{cases} \theta_1(\mathrm{t}) = \dfrac{\mu}{m_2}\theta_0 cos(\omega_1 t) + \dfrac{\mu}{m_1}\theta_0 cos(\omega_2 t) \\ \theta_2(\mathrm{t}) = \dfrac{\mu}{m_2}\theta_0 cos(\omega_1 t) - \dfrac{\mu}{m_2}\theta_0 cos(\omega_2 t) \end{cases}$$

Exercise №2: (inertial coupling)

The system shown in the figure below consists of two identical simple pendulums of mass $m$and length l placed end to end.



1. Calculate the exact Lagrange function of the system.
2. Give the approximate expression of the Lagrange function in the case of small amplitude oscillations.
3. Establish the differential equations of motion in θ_1 and θ_2.
4. Calculate the propre pulsations ω_1 et ω_2 and give the two associated eigenmodes, as well as the general solution of the movement of the system.
5. Find the simplest initial conditions to achieve for the system to oscillate according to the first vibration mode. Same question for the second mode.

Solution (Exercise №2):

1. the exact Lagrange function of the system: $\mathcal{L} = \mathrm{T} - \mathrm{U}$

- kinetic energy: $\mathrm{T} = \mathrm{T}_{m1} + \mathrm{T}_{m2}$

$$\begin{cases} \mathrm{T}_{m1} = \dfrac{1}{2} m_1 {v_1}^2 \\ \\ \mathrm{T}_{m2} = \dfrac{1}{2} m_2 {v_1}^2 \end{cases}$$

$$\overrightarrow{om_1} = lsin\theta_1\vec{\imath} + lcos\theta_1\vec{\jmath} \Rightarrow \vec{v}_1 = \frac{d\overrightarrow{om_1}}{dt}$$

$$\Rightarrow \vec{v}_1 = l\,\dot{\theta}_1 cos\theta_1\vec{\imath} - l\,\dot{\theta}_1 sin\theta_1\vec{\jmath} \Rightarrow v_1 = l\,\dot{\theta}_1$$
$$\mathrm{T}_{m1} = \frac{1}{2}m_1 l^2\,\dot{\theta}_1{}^2$$

$$\overrightarrow{om_2} = \overrightarrow{om_1} + \overrightarrow{m_1m_2} = (lsin\theta_1\vec{\imath} + lcos\theta_1\vec{\jmath}) + (lsin\theta_2\vec{\imath} + lcos\theta_2\vec{\jmath}) \Rightarrow \vec{v}_1 = \frac{d\overrightarrow{om_2}}{dt}$$

$$\Rightarrow \vec{v}_1 = l\,\dot{\theta}_1(cos\theta_1\vec{\imath} - sin\theta_1\vec{\jmath}) + l\,\dot{\theta}_2(cos\theta_2\vec{\imath} - sin\theta_2\vec{\jmath})$$

$$\Rightarrow \vec{v}_1 = \left(l\,\dot{\theta}_1 cos\theta_1 + l\,\dot{\theta}_2 cos\theta_2\right)\vec{\imath} + \left(l\,\dot{\theta}_1 sin\theta_1 + l\dot{\theta}_2 sin\theta_2\right)\vec{\jmath}$$
$$\Rightarrow v_2{}^2 = l^2\,\dot{\theta}_1{}^2 + l^2\,\dot{\theta}_2{}^2 + 2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_1)$$
$$\mathrm{T}_{m2} = \frac{1}{2}m_2 l^2\left(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_1)\right)$$
$$\mathrm{T} = \mathrm{T}_{m1} + \mathrm{T}_{m2} \Rightarrow \mathrm{T} = \frac{1}{2}m_1 l^2\,\dot{\theta}_1{}^2 + \frac{1}{2}m_2 l^2\left(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_1)\right)$$
$$m_1 = m_2 = m \Rightarrow \mathrm{T} = \frac{1}{2}ml^2\left(2\,\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_1)\right)$$

- potential energy: $U = U_{m1} + U_{m2}$

$$\begin{cases} U_{m1} = mgl(1 - cos\theta_1) \\ U_{m2} = mgl(2 - cos\theta_1 + cos\theta_2) \end{cases} \Rightarrow U = mgl(1 - cos\theta_1) + mgl(2 - cos\theta_1 + cos\theta_2)$$

the exact Lagrange function: $\mathcal{L} = \mathrm{T} - \mathrm{U}$
$$\Rightarrow \mathcal{L} = \frac{1}{2}ml^2\left(2\,\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_1)\right) - mgl(1 - cos\theta_1) + mgl(2 - cos\theta_1 + cos\theta_2)$$

2- the approximate expression of the Lagrange function:
   In the case of small amplitude:

$$(\theta_1 \ll, \theta_2 \ll) \Rightarrow \begin{cases} cos\theta \approx 1 - \frac{\theta^2}{2} \\ sin\theta \approx \theta \\ \theta_1 - \theta_2 \cong 0 \end{cases}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}ml^2\left(2\,\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\right) - \frac{1}{2}mgl(2\theta_1{}^2 + \theta_2{}^2)$$

3- Establish the differential equations of motion in θ_1 and θ_2:

The Lagrange equation's: $\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = 0 \end{cases}$

$$\Rightarrow \begin{cases} \frac{1}{2}ml^2(4\ddot{\theta}_1 + 2\ddot{\theta}_2) + \frac{1}{2}mgl\,(4\theta_1) = 0 \\ \frac{1}{2}ml^2(2\ddot{\theta}_2 + 2\ddot{\theta}_1) + \frac{1}{2}mgl\,(2\theta_2) = 0 \end{cases} \Rightarrow \begin{cases} \ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{1}{2}\ddot{\theta}_2 = 0 \\ \ddot{\theta}_2 + \frac{g}{l}\theta_2 + \ddot{\theta}_1 = 0 \end{cases}$$

4- the propre pulsations ω_1 et ω_2:

The solution:

$\theta_1(t) = A_1 cos(\omega t + \varphi) \Rightarrow \overline{\theta_1} = \overline{A_1} e^{j\omega t}$

$\theta_2(t) = A_2 cos(\omega t + \varphi) \Rightarrow \overline{\theta_2} = \overline{A_2} e^{j\omega t}$

By replacing in the equations: $\Rightarrow \begin{cases} (\frac{g}{l} - \omega^2)A_1 - \frac{1}{2}\omega^2 A_2 = 0 \\ -\omega^2 A_1 + (\frac{g}{l} - \omega^2)A_2 = 0 \end{cases}$

We can write in matrix form as follows: $\begin{pmatrix} (\frac{g}{l} - \omega^2) & -\frac{1}{2}\omega^2 \\ -\omega^2 & (\frac{g}{l} - \omega^2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The metrically system admits a non-zero solution if: $det \begin{vmatrix} (\frac{g}{l} - \omega^2) & -\frac{1}{2}\omega^2 \\ -\omega^2 & (\frac{g}{l} - \omega^2) \end{vmatrix} = 0$

$\Rightarrow (\frac{g}{l} - \omega^2)^2 - \frac{1}{2}\omega^4 = 0$

$\Rightarrow \frac{1}{2}\omega^4 - 2\frac{g}{l}\,\omega^2 + (\frac{g}{l})^2 = 0 \quad \Rightarrow \omega^4 - 4\frac{g}{l}\,\omega^2 + 2(\frac{g}{l})^2 = 0$

$$\begin{cases} \omega_1 = \sqrt{2 - \sqrt{2}}\;\frac{g}{l} \\ \omega_2 = \sqrt{2 + \sqrt{2}}\;\frac{g}{l} \end{cases} \Rightarrow \begin{cases} \omega_1 = 0.765\,\frac{g}{l} \\ \omega_2 = 1.848\,\frac{g}{l} \end{cases}$$

5- Find the simplest initial conditions to achieve for the system to oscillate according to the first vibration mode. Same question for the second mode:

1st mode of vibration: for $\omega_1 = 0.765\,\frac{g}{l}$

$$-\omega^2 A_1 + \left(\frac{g}{l} - \omega^2\right)A_2 = 0 \Rightarrow \frac{A_1}{A_2} = \frac{\frac{g}{l} - {\omega_1}^2}{{\omega_1}^2} = 0.859 > 0 \Rightarrow \varphi_1 = \varphi_2$$

The masses vibrate in phase:

$\theta_1(t) = A_1 cos(\omega t + \varphi_1)$

$\theta_2(t) = 0.859 A_1 cos(\omega t + \varphi_1)$

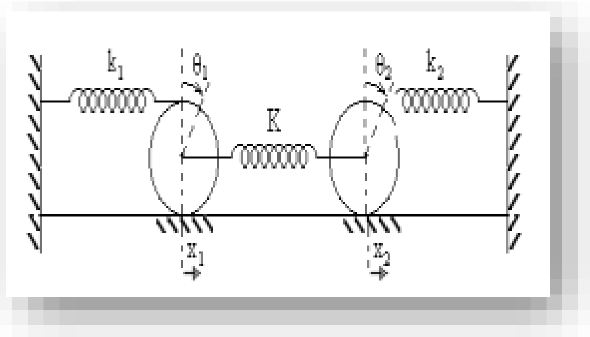
1nd mode of vibration: for $\omega_1 = 1.848\,\frac{g}{l}$

$$-\omega^2 A_1 + \left(\frac{g}{l} - \omega^2\right)A_2 = 0 \Rightarrow \frac{A_1}{A_2} = \frac{\frac{g}{l} - {\omega_1}^2}{{\omega_1}^2} = -0.718 > 0 \Rightarrow \varphi_2 = \varphi_1 + \pi$$

The masses vibrate in opposite phase:

$\theta_1(t) = A_1 cos(\omega t + \varphi_1)$

$\theta_2(t) = 0.718 A_1 cos(\omega t + \varphi_1 + \pi)$



<u>Exercise №3:</u>

The following mechanical oscillatory system is considered:

The 2 cylinders of the same mass M and the same radius R roll without slipping, that is to say that when they rotate respectively by θ_1 and θ_2, their centers of gravity move respectively by $x_1 = R\theta_1$ and $x_2 = R\theta_2$

1. Determine the kinetic energy and potential energy of the system (θ_1 and θ_2 will be chosen as variables)
2. Establish the Lagrange function for $k_1 = k_2 = k$ and
   $K = 2k$. From this deduce the system of differential equations of motion
3. Find the propre pulsations corresponding to the possible vibration modes.
4. Deduce the passage matrix and write the general solutions.

التمرين الثالث:

يعتبر نظام الاهتزاز الميكانيكي التالي: الأسطوانتان لهما نفس الكتلة M ونفس نصف القطر R تتدحرجان دون انزلاق، أي أنهما عندما تدوران على التوالي بمقدار θ_1 و θ_2، يتحرك مراكز ثقلهما على التوالي بمقدار $x_1 = R\theta_1$ و $x_2 = R\theta_2$

1. جد الطاقة الحركية والطاقة الكامنة للنظام بدلالة θ_1 و θ_2
2. أعطي دالة لاغرونج في حالة $K = 2k$ و $k_1 = k_2 = k$. ثم استنتج المعادلات التفاضلية للحركة
3. ابحث عن النبضين الذاتيين المقابلين لأنماط الاهتزاز المحتملة.