Chapter 5: Linear systems with tow degrees of freedom

5.1 Systems with two degrees of freedom:

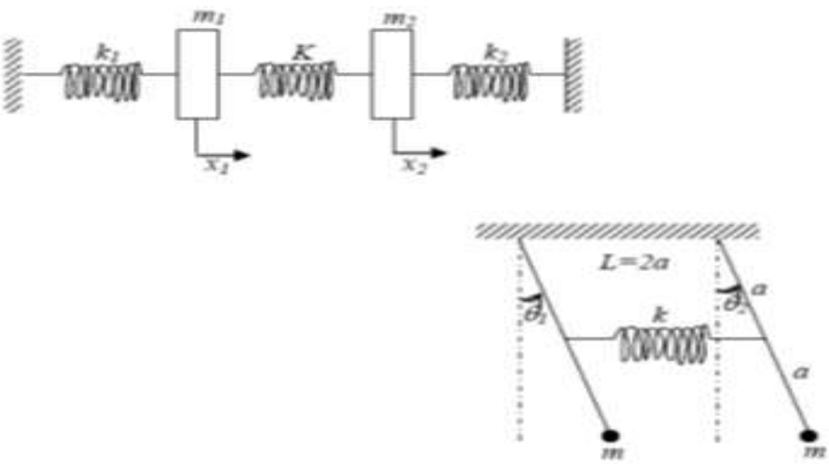
A system with two degrees of freedom can be considered as the coupling of two systems with a single degree of freedom. Therefore, we need two differential equations obtained from a system of two Lagrange equations

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_1}\right) + \dfrac{\partial D_a}{\partial \dot{q}_1} - \dfrac{\partial L}{\partial q_1} = F_{q_1} \\ \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_2}\right) + \dfrac{\partial D_a}{\partial \dot{q}_2} - \dfrac{\partial L}{\partial q_2} = F_{q_2} \end{cases}$$

5.2 Types of coupling:

a- Elastic coupling:

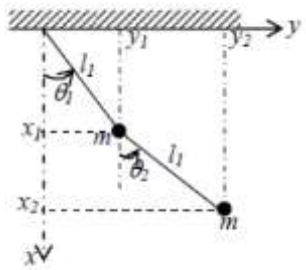
The coupling between the two systems is through a spring.



Elastic coupling

b- Inertial coupling:

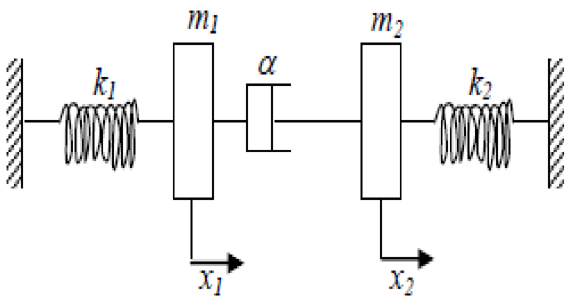
The coupling between the two systems is through a mass.



Inertial coupling

c-Viscous coupling:

The coupling between the two systems is through a damper.



Viscous coupling

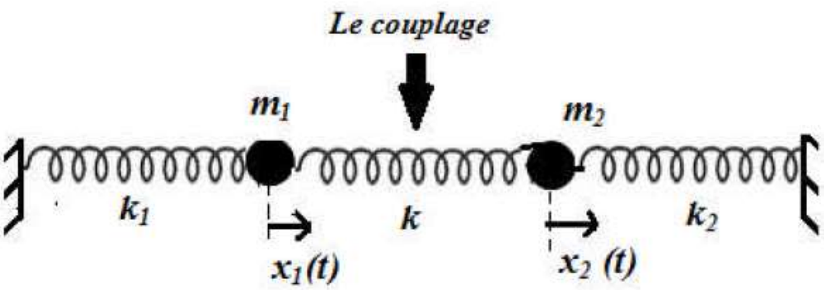
5.3 Free and undamped oscillation:

In the case of free and undamped oscillations, the Lagrange equations ($\boldsymbol{F_q} = \mathbf{0}$ *and* $\boldsymbol{D_a} = \mathbf{0}$) are reduced to:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q_1}}\right) - \dfrac{\partial L}{\partial q_1} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q_2}}\right) - \dfrac{\partial L}{\partial q_2} = 0 \end{cases}$$

5.4 Equation of motion:

Consider the free opposite system. The two independent variables are x_1 and x_2. k is called the coupling element.



- The kinetic energy is written as follows:$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$
- The potential energy:$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k(x_1 - x_2)^2$
- Lagrange function is written:

$$L = T - U \Rightarrow L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k(x_1 - x_2)^2$$

$$\Rightarrow L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k(x_1^2 + x_2^2 - 2x_1x_2)$$

$$\Rightarrow L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2x_2^2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2-\frac{1}{2}k(-2x_1x_2)$$

$$\Rightarrow L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2-\frac{1}{2}k_1x_1^2-\frac{1}{2}k_2x_2^2-\frac{1}{2}kx_1^2-\frac{1}{2}kx_2^2+kx_1x_2$$

- The two Lagrange equations are written: ($For\ D_a=0, F=0:$ $system\ non\ damped\ and\ non\ forced$)

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right)-\left(\dfrac{\partial L}{\partial x_1}\right)=0\\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right)-\left(\dfrac{\partial L}{\partial x_2}\right)=0\end{cases}$$

The coupled differential system then becomes:

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right)-\left(\dfrac{\partial L}{\partial x_1}\right)=0\\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right)-\left(\dfrac{\partial L}{\partial x_2}\right)=0\end{cases}\Rightarrow\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_1}\right)=m_1\ddot{x}_1,\left(\dfrac{\partial L}{\partial x_1}\right)=-k_1x_1-kx_1+kx_2\\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_2}\right)=m_2\ddot{x}_2,\left(\dfrac{\partial L}{\partial x_2}\right)=-k_2x_2-kx_2+kx_1\end{cases}$$

$$\Rightarrow\begin{cases}m_1\ddot{x}_1+(k_1+k)x_1-kx_2=0\\ m_2\ddot{x}_2+(k+k_2)x_2-kx_1=0\end{cases}\Rightarrow\begin{cases}\ddot{x}_1+\dfrac{(k_1+k)}{m_1}x_1-\dfrac{k}{m_1}x_2=0\\ \ddot{x}_2+\dfrac{(k+k_2)}{m_2}x_2-\dfrac{k}{m_2}x_1=0\end{cases}$$

5.5 the propre modes:

For the system of two differential equations to admit a solution, it is necessary that $x_1(t)$ and $x_2(t)$ are two harmonic functions of the same pulsationω, but of different amplitudes $A_1$and A_2:

$$\boldsymbol{x_1(t)=A_1cos(\omega t+\varphi)\Rightarrow \ddot{x}_1(t)=-A_1\omega^2cos(\omega t+\varphi)}$$

$$\boldsymbol{x_2(t)=A_2cos(\omega t+\varphi)\Rightarrow \ddot{x}_2(t)=-A_2\omega^2cos(\omega t+\varphi)}$$

To find ω, use the complex representation:

$$\Rightarrow\begin{cases}x_1=A_1\cos(\omega t+\varphi_1)\rightarrow\overline{x_1}=A_1e^{j(\omega t+\varphi_1)}=\overline{A}_1e^{j\omega t}\\ x_2=A_2\cos(\omega t+\varphi_2)\rightarrow\overline{x_2}=A_2e^{j(\omega t+\varphi_2)}=\overline{A}_2e^{j\omega t}\end{cases}$$

$$\Rightarrow\begin{cases}\overline{x_1}=\overline{A}_1e^{j\omega t}\\ \overline{x_2}=\overline{A}_2e^{j\omega t}\end{cases}=\Rightarrow\begin{cases}\overline{\dot{x}_1}=j\omega\overline{A}_1e^{j\omega t}\\ \overline{\dot{x}_2}=j\omega\overline{A}_2e^{j\omega t}\end{cases}\Rightarrow\begin{cases}\overline{\ddot{x}}_1=(j\omega)^2\overline{A}_1e^{j\omega t}=-\omega^2\overline{A}_1e^{j\omega t}\\ \overline{\ddot{x}}_2=(j\omega)^2\overline{A}_2e^{j\omega t}=-\omega^2\overline{A}_2e^{j\omega t}\end{cases}$$

$$\Rightarrow\begin{cases}\ddot{x}_1+\dfrac{(k_1+k)}{m_1}x_1-\dfrac{k}{m_1}x_2=0\\ \ddot{x}_2+\dfrac{(k+k_2)}{m_2}x_2-\dfrac{k}{m_2}x_1=0\end{cases}\Rightarrow\begin{cases}-\omega^2\overline{A}_1e^{j\omega t}+\dfrac{(k_1+k)}{m_1}\overline{A}_1e^{j\omega t}-\dfrac{k}{m_1}\overline{A}_2e^{j\omega t}=0\\ -\omega^2\overline{A}_2e^{j\omega t}+\dfrac{(k+k_2)}{m_2}\overline{A}_2e^{j\omega t}-\dfrac{k}{m_2}\overline{A}_1e^{j\omega t}=0\end{cases}$$

$$\begin{cases} -\omega^2\overline{A}_1 e^{j\omega t} + \dfrac{(k_1+k)}{m_1}\overline{A}_1 e^{j\omega t} - \dfrac{k}{m_1}\overline{A}_2 e^{j\omega t} = 0 \\ -\omega^2\overline{A}_2 e^{j\omega t} + \dfrac{(k+k_2)}{m_2}\overline{A}_2 e^{j\omega t} - \dfrac{k}{m_2}\overline{A}_1 e^{j\omega t} = 0 \end{cases} \Rightarrow \begin{cases} \left(-\omega^2 + \dfrac{(k_1+k)}{m_1}\right)\overline{A}_1 - \left(\dfrac{k}{m_1}\right)\overline{A}_2 = 0 \\ \left(-\omega^2 + \dfrac{(k+k_2)}{m_2}\right)\overline{A}_2 - \left(\dfrac{k}{m_2}\right)\overline{A}_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left(-\omega^2 + \dfrac{(k_1+k)}{m_1}\right)\overline{A}_1 - \left(\dfrac{k}{m_1}\right)\overline{A}_2 = 0 \\ -\left(\dfrac{k}{m_2}\right)\overline{A}_1 + \left(-\omega^2 + \dfrac{(k+k_2)}{m_2}\right)\overline{A}_2 = 0 \end{cases} \Rightarrow \begin{cases} (-\omega^2 + a)\overline{A}_1 - b\overline{A}_2 = 0 \\ -c\overline{A}_1 + (-\omega^2 + d)\overline{A}_2 = 0 \end{cases}$$

$$a = \frac{k+k_1}{m_1},\ b = \frac{k}{m_1},\ c = \frac{k}{m_2},\ d = \frac{k+k_2}{m_2}.$$

We can write in matrix form as follows:

$$\begin{pmatrix} (\boldsymbol{a} - \boldsymbol{\omega}^2) & -\boldsymbol{b} \\ -\boldsymbol{c} & (\boldsymbol{d} - \boldsymbol{\omega}^2) \end{pmatrix} \begin{pmatrix} \overline{\boldsymbol{A_1}} \\ \overline{\boldsymbol{A_2}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$$

For the equation to be true without $\overline{A_1}$ and $\overline{A_2}$ both being zero, its characteristic determinant must be zero:

$$det = \begin{vmatrix} (a - \omega^2) & -b \\ -c & (d - \omega^2) \end{vmatrix} = (a - \omega^2)(d - \omega^2) - (-c)(-b) = 0$$

$$\Rightarrow det = \omega^4 - \omega^2 d - \omega^2 a + ad - bc = \omega^4 - (d + a)\omega^2 + (ad - bc) = 0$$

This gives us the characteristic equation: $\omega^4 - (d + a)\omega^2 + (ad - bc) = 0$

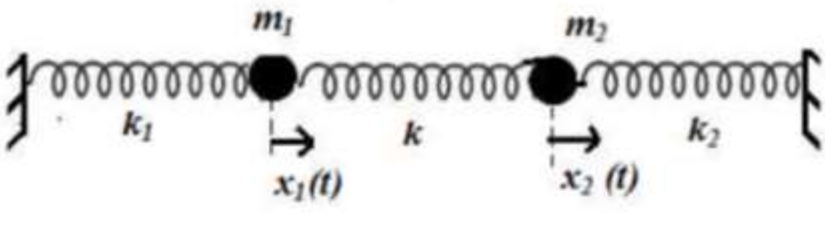
The two real and positive solutions ω_1 and ω_2 of this equation are called proper or normal pulsations. The smaller one is called the fundamental, the other is called the harmonic.

- *First propre mode*: For $\omega=\omega_1$, the system implies that: $\frac{A_{1(1^{st}\,mode)}}{A_{2(1^{st}\,mode)}} = \frac{d-\omega_1^2}{c} > \boldsymbol{0}$

.

The solution is written in this case:

$$\begin{cases} \boldsymbol{x}_{\boldsymbol{1}(\boldsymbol{1}^{st}\,\boldsymbol{mode})}(\boldsymbol{t}) = \boldsymbol{A}_{\boldsymbol{1}(\boldsymbol{1}^{st}\,\boldsymbol{mode})}\boldsymbol{cos}(\boldsymbol{\omega_1 t} + \boldsymbol{\varphi}) \\ \boldsymbol{x}_{\boldsymbol{2}(\boldsymbol{1}^{st}\,\boldsymbol{mode})}(\boldsymbol{t}) = \boldsymbol{A}_{\boldsymbol{2}(\boldsymbol{1}^{st}\,\boldsymbol{mode})}\boldsymbol{cos}(\boldsymbol{\omega_1 t} + \boldsymbol{\varphi}) \end{cases}$$

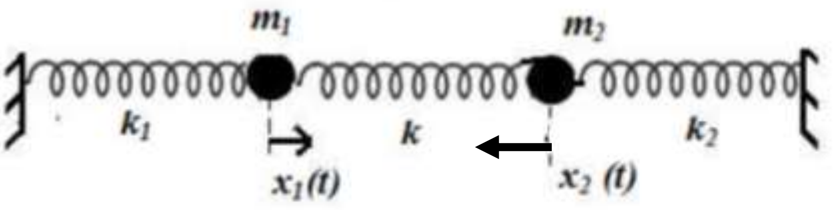


1^st propre mode

- *Second propre mode*: For $\omega=\omega_2$, the system implies that: $\frac{A_{1(2^{nd}\,mode)}}{A_{2(2^{nd}\,mode)}} = \frac{d-\omega_1^2}{c} < 0$

The solution is written in this case:

$$\begin{cases} x_{1(2^{nd}\,mode)}(t) = A_{1(2^{nd}\,mode)}cos(\omega_1 t + \varphi) \\ x_{2(2^{nd}\,mode)}(t) = -A_{2(2^{nd}\,mode)}cos(\omega_1 t + \varphi) \end{cases}$$



2nd propre mode