4.3. Methods of resolution:

4.3.1. Mode Superposition Method

The method is applied in two steps. First, the eigenvalues (frequencies) and eigenvectors (mode shapes) of the homogeneous equation:

$$[\boldsymbol{K}]\{\boldsymbol{U}\}+(\boldsymbol{\alpha}[\boldsymbol{K}]+\boldsymbol{\beta}[\boldsymbol{M}])\{\dot{\boldsymbol{U}}\}+[\boldsymbol{M}]\{\ddot{\boldsymbol{U}}\}=\mathbf{0} \qquad (4.1)$$

are determined. These modes are then superposed to solve forced vibration problems.

4.3.1.1. Free Vibrations

in the case of undamped vibrations, $[C]=0$*, the homogeneous differential equation is written as:*

$$[\boldsymbol{K}]\{\boldsymbol{U}\}+[\boldsymbol{M}]\{\ddot{\boldsymbol{U}}\}=\mathbf{0} \qquad (4.2)$$

To find the natural frequencies (eigenvalues) and the principal or fundamental modes (eigenvectors),

Let's express the nodal vector $\{U(t)\}$ *as:*

$$\{\boldsymbol{U}(\boldsymbol{t})\}=\{\emptyset\}\boldsymbol{e}^{\boldsymbol{i\omega t}} \qquad (4.3)$$

Where: $\{\emptyset\}$ *is the unknown vector of amplitudes and* ω *(rad/sec) is the natural frequency,*

the last equation can be written in the form:

$$[\boldsymbol{K}]\{\emptyset\}\boldsymbol{e}^{\boldsymbol{i\omega t}}-\boldsymbol{\omega}^{2}[\boldsymbol{M}]\{\emptyset\}\boldsymbol{e}^{\boldsymbol{i\omega t}}=\mathbf{0}$$

$$([\boldsymbol{K}]-\boldsymbol{\omega}^{2}[\boldsymbol{M}])\{\emptyset\}=\mathbf{0} \qquad (4.4)$$

Since that $[K]$ *and* $[M]$ *are of order* $n \times n$*, this equation provides n eigenvalues* $\omega^2{}_r$ $(r\colon 1,2,\dots,n)$*. Each frequency* ω_r *corresponds to a modal shape* $\{(\varphi\}_r$ $(r = 1,2,\dots,n)$*. The frequencies* ω_r *are positive if* $[K]$ *and* $[M]$ *are positive definite, which is generally the case.*

It is rarely practical to write out the polynomial expansion of the determinant for large matrices. Several standard computer programs are available in open literature as library subroutines for this purpose. In many cases, only a few modes $\{(\varphi\}_r$ $(r = 1,2,\dots,p, p \leq n)$ *are needed. Several iterative solution procedures allow for solving these modes (Clough and Penzien, 1975; Bathe, 2001).*

An important property of modal shapes, which is very useful for solving forced vibration problems, is their orthogonality with respect to the mass matrix $[M]$. *The orthogonality condition can be formulated as follows:*

$\{\emptyset\}_r^T[\boldsymbol{M}]\{\emptyset\}_s = \begin{cases} \boldsymbol{0} \ \boldsymbol{r} \neq \boldsymbol{s} \\ \boldsymbol{M_r} \ \boldsymbol{r} = \boldsymbol{s} \end{cases}$ *(4.5), where* $\{\emptyset\}_r$ *and* $\{\emptyset\}_s$ *are mode shapes corresponding to frequencies* ω_r *and* ω_s *respectively.*

The orthogonality relation for the stiffness matrix [K] *can be established from the equation:* $([\boldsymbol{K}] - \boldsymbol{\omega^2}[\boldsymbol{M}])\{\emptyset\} = \boldsymbol{0}$ *as*

$$K_r = \{\varphi\}_r^T[K]\varphi_r = \omega_r^2\{\varphi\}_r^T[M]\{\varphi\}_r = \omega_r^2 M_r \quad (4.6)$$

$$\omega_r^2 = \frac{K_r}{M_r}$$

M_r *and* K_r *are called generalized mass and stiffness, respectively and* ω_r *is the natural frequency of the rth mode.*

Sometimes the mode shapes are normalized as $\{\hat{\varphi}\}_r = 1/\sqrt{M_r}\{\varphi\}$ *which makes the generalized mass* $\widehat{M}_r = 1$ *and the generalized stiffness* $\widehat{K}_r = \omega_r^2$

In the next section, only the mode shapes {φ}r will be employed to solve transient problems.

4.3.1.2 Transient Response

The orthogonality properties (Eqs 4.5 and 4.6) of the mode shapes now may be used to uncouple the non-homogeneous equation of motion (4.0), The solution vector {q(t)} is expressed by superposing the modes as

$\{\boldsymbol{Q}(\boldsymbol{t})\}_{\boldsymbol{n\times1}} = [\boldsymbol{\Phi}]_{\boldsymbol{n\times p}}\{\boldsymbol{Y}(\boldsymbol{t})\}_{\boldsymbol{p\times1}}$ *(4.7) Where the modal matrix:*

$$[\boldsymbol{\Phi}] = \left[\{\boldsymbol{\varphi}\}_1, \{\boldsymbol{\varphi}\}_2, \dots, \{\boldsymbol{\varphi}\}_r \ \dots\dots \{\boldsymbol{\varphi}\}_p\right] (\boldsymbol{p} \leq \boldsymbol{n})$$

and {Y(t)} is the vector of unknown modal amplitudes. Note that [Φ] transforms the generalized coordinates {Y(t)} to the physical coordinates {Q(t)}.

By substituting Eq. 4.7 *in Eq.* 4.0 *and then pre-multiplying the resulting Eq. by [Φ]*T*, the following equation set is obtained.*

$$[\boldsymbol{\Phi}]^T[\boldsymbol{M}][\boldsymbol{\Phi}]\{\ddot{\boldsymbol{Y}}\} + [\boldsymbol{\Phi}]^T[\boldsymbol{K}][\boldsymbol{\Phi}]\{\boldsymbol{Y}\} = [\boldsymbol{\Phi}]^T\{\boldsymbol{F}\} \ (\boldsymbol{4.8})$$

Using the orthogonality Eqs. (4.5) and (4.6), Eq. 4.8 reduces to a set of uncoupled ordinary differential equations shown below.

$$\boldsymbol{M_r\ddot{Y}_r + K_rY_r = F_r}$$

where the generalized force $\boldsymbol{Fr(t)} = \{(\boldsymbol{\varphi}\}_r^T\{\boldsymbol{F(t)}\}$. *Thus, the problem statement can be stated as*

$$\ddot{\boldsymbol{Y}}_r + \boldsymbol{\omega}_r^2 \boldsymbol{Y}_r = \boldsymbol{R}_r(\boldsymbol{t}), \boldsymbol{where}\ \boldsymbol{R}_r(\boldsymbol{t}) = \frac{\boldsymbol{F}_r(\boldsymbol{t})}{\boldsymbol{M}_r} \quad (\boldsymbol{r} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{p};\ \boldsymbol{p} \leq \boldsymbol{n}) \quad \textbf{(4.9)}$$

Initial conditions are

$$\{Y\}_r\Big|_{t=0} = \frac{1}{M_r}\{\varphi\}_r^T[M]\{q(0)\}$$

$$\{\dot{\boldsymbol{Y}}\}_r\Big|_{\boldsymbol{t}=\boldsymbol{0}} = \frac{\boldsymbol{1}}{\boldsymbol{M}_r}\{\boldsymbol{\varphi}\}_r^T[\boldsymbol{M}]\{\dot{\boldsymbol{q}}(\boldsymbol{0})\} \quad (4.10)$$

It can be noted that Eq. (4.9) is a governing differential equation for a single-dof system.

Solution can be obtained using the Duhamel integral as

$$\boldsymbol{Y}_r(\boldsymbol{t}) = \frac{\boldsymbol{1}}{\boldsymbol{\omega}_r}\int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{R}_r(\boldsymbol{\tau}) \boldsymbol{sin}\ \boldsymbol{\omega}_r(\boldsymbol{t}-\boldsymbol{\tau})\boldsymbol{d\tau} + \boldsymbol{C}_1 \boldsymbol{sin}\ \boldsymbol{\omega}_r \boldsymbol{t} + \boldsymbol{C}_2 \cos \boldsymbol{\omega}_r \boldsymbol{t} \quad (4.11)$$

where the constant C*1 and* C*2 are to be determined from the initial conditions. The integral in the above equation may have to be evaluated numerically.*

Once all p *solutions of Eq. (4.9) are determined, the complete response is obtained by superposing the modes as follows.*

$$\{\boldsymbol{Q}(\boldsymbol{t})\} = \sum_{\boldsymbol{r}=\boldsymbol{1}}^{\boldsymbol{p}} \{\boldsymbol{\varphi}\}_r \boldsymbol{Y}_r(\boldsymbol{t}) \quad (4.12)$$

The mode superposition method is favored if only a few modes are needed to describe the response. For example, an earthquake usually has only a few dominant frequencies in its frequency spectra. However, the method has also been used for a concentrated force (Green's function) in wave propagation problems (Datta and Shah, 2008), employing few modes p *(*$p < n$*).*