Chapter 4: FINITE ELEMENTS IN DYNAMICS

4.1. Introduction

In previous chapters, we assumed, when constructing finite element equations $\{\boldsymbol{f}\} = [\boldsymbol{k}]\{\boldsymbol{d}\}$, that the loads applied to elements or the entire structure are static loads that do not vary over time. These equations are sufficiently accurate if the load $\{\boldsymbol{f}\}$ has a frequency lower than one-quarter of the lowest natural frequency of the structure. However, the frequencies of loads caused by moving vehicles, machinery, earthquakes, explosions, etc., are quite high, and a dynamic analysis to account for the effects of inertial forces then becomes necessary.

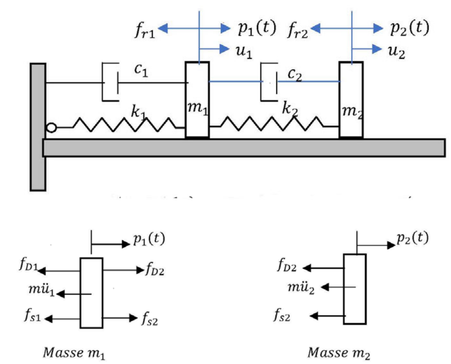
4.2 Formulation of the Dynamic Problem

4.2.1 Direct approach:

To present the direct approach, let us use the linear spring system shown in the figure below, where:

$p_1(t)$ *and* $p_2(t)$ *are the dynamic forces applied to the masses* m_1 *and* m_2

f_{r1} *and* f_{r2} *are the forces resisting the displacements* u_1 *and* u_2 *of the masses* m_1 *and* m_2



k_1 et k_2 are the stiffness coefficients of the springs, and c_1, c_2 are the damping factors

The forces resisting motion are:

- *The elastic restoring force of the spring:* $f_s = k(u_2 - u_1)$
- *The damping force:* $f_D = c(\dot{u}_2 - \dot{u}_1)$
- *The inertial mass force:* $-m\ddot{u}$

The equilibrium of forces gives:

For mass m_1 $: -f_{s1} - f_{D1} - m_1\ddot{u}_1 + f_{s2} + f_{D2} + p_1(t) = 0$
For mass m_2 $-f_{s2} - f_{D2} - m_2\ddot{u}_2 + p_2(t) = 0$

Or

$$f_{s1} = k_1(u_1 - 0) = k_1 u_1\,; \quad f_{s2} = k_2(u_2 - u_1)$$
$$f_{D1} = c_1(\dot{u}_1 - 0) = c_1\dot{u}_1\,; \; f_{D2} = c_2(\dot{u}_2 - \dot{u}_1)$$

The equilibrium is then written as:

$$-k_1u_1 - c_1\dot{u}_1 - m_1\ddot{u}_1 + k_2(u_2 - u_1) + c_2(\dot{u}_2 - \dot{u}_1) + p_1(t) = 0$$
$$-k_2(u_2 - u_1) - c_2(\dot{u}_2 - \dot{u}_1) - m_2\ddot{u}_2 + p_2(t) = 0$$

Which gives:

$$(k_1 + k_2)u_1 - k_2u_2 + (c_1 + c_2)\dot{u}_1 - c_2\dot{u}_2 + m_1\ddot{u}_1 = p_1(t)$$
$$-k_2u_1 + k_2u_2 - c_1\dot{u}_1 + c_2\dot{u}_2 + m_2\ddot{u}_2 = p_2(t)$$

These two equations can be written in the following matrix form:

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Or in condensed form:

$$[K]\{D\} + [C]\{\dot{D}\} + [M]\{\ddot{D}\} = \{F\}$$

Where:

$[K]$, $[C]$, $[M]$, $\{F\}$, $\{D\}$, $\{\dot{D}\}$ *et* $\{\ddot{D}\}$ *are, respectively, the stiffness matrix, the damping matrix, the mass matrix, the vector of applied dynamic forces, the vector of mass displacements, the velocity vector, and the acceleration vector for the entire spring system.*

4.2.2 Formulation by the Principle of Virtual Work

Given a structure discretized into "m" elements, and approximating the displacement U of an element by: $[N]\{d\}$, *the force vector* $\{f^e\}$ *of the element will be given by:*

$$\{\boldsymbol{f^e}\} = \int_V [\boldsymbol{N}]^T\{\boldsymbol{f_b}\}\,\boldsymbol{dv} + \int_S [\boldsymbol{N}]^T\{\boldsymbol{f_s}\}\,\boldsymbol{ds} + \{\boldsymbol{f_n}\}$$

If one or more of these forces are dynamic in nature, taking into account the inertia and damping forces created, the force vector at the nodes of the element is expressed as follows:

$$\{f^e\} = \int_V [N]^T\{f_b\}dv - \int_V [\bar{c}][N]^T\{\dot{U}\}dv - \int_V [\rho][N]^T\{\ddot{U}\}dv + \int_S [N]^T\{f_s\}\,ds + \{f_n\}$$

$$\{f^e\} = \int_V [N]^T\{f_b\}dv - \int_V [\bar{c}][N]^T[N]\{\dot{d}\}dv - \int_V [\rho][N]^T[N]\{\ddot{d}\}dv + \int_S [N]^T\{f_s\}\,ds + \{f_n\}$$

In this last equation:

- $\int_V [N]^T\{f_b\}dv$ *and* $\int_S [N]^T\{f_s\}\,ds$ *are the vector of forces equivalents to the volume loads* $\{f_b\}$ *and the surface loads* $\{f_s\}$,

$\int_V [\bar{c}][N]^T[N]\{\dot{d}\}dv$ *Is the vector of loads equivalent to the damping forces* $[\bar{c}]\dot{U}$ *per unit volume of the element and*

$\int_V [\rho][N]^T[N]\{\ddot{d}\}dv$ *is the vector of loads equivalent to the inertial force* -$[\rho]\ddot{U}$ *where* $[\rho]$ *is the mass per unit volum*

Therefore:

$$\{f^e\} = \int_V [N]^T\{f_b\}dv + \int_S [N]^T\{f_s\}\,ds + \{f_n\} - \int_V [\bar{c}][N]^T[N]\{\dot{d}\}dv - \int_V [\rho][N]^T[N]\{\ddot{d}\}dv$$

$$\{f^e\} = [k]\{d\} \rightarrow [k]\{d\} = \{f_n\} + \{f_{eqst}\} + \{f_{eqdyn}\}$$

Where:

- $\{f_n\}$*: loads applied to the nodes of the element*
- $\{f_{eqst}\}$*: loads equivalent to the non-nodal loads*
- $\{f_{eqdyn}\}$*: loads equivalent to the inertial forces and the damping forces*

Rewriting $\{f_{eqdyn}\}$ *in the form:*

$$\{f_{eqdyn}\} = \int_V [\bar{c}][N]^T[N]\{\dot{d}\}dv - \int_V [\rho][N]^T[N]\{\ddot{d}\}dv = -[c]\{\dot{d}\} - [m]\{\ddot{d}\}$$

The dynamic equilibrium equation takes the form

$$[k]\{d\} + [c]\{\dot{d}\} + [m]\{\ddot{d}\} = \{f(t)\}$$

where:

- $[m]$ *is the coherent mass matrix of the element* $[m] = \int_V [\rho][N]^T[N]dv$
- $[c]$ *is the coherent damping matrix of the element* $[c] = \int_V [c][N]^T[N]dv$
- $[k]$ *is the stiffness matrix of the element* $[c] = \int_V [B]^T[E][B]dv$*, Because the damping matrix is noted* $[C]$*, we have noted here by* $[E]$ *the behavior matrix previously also noted* $[C]$
- $\{f(t)\}$ *is the vector of loads applied to the element*

The assembly of the elements gives the global dynamic equilibrium equation:

$$[K]\{D\} + [C]\{\dot{D}\} + [M]\{\ddot{D}\} = \{F(t)\}$$

In practice, determining the damping matrix $[C]$ *is difficult and little known. The matrix* $[C]$ *is calculated as a combination of the matrices* $[K]$ *and* $[M]$*:*

$$[C] = \alpha[K] + \beta[M]$$

Where α *and* β *are determined experimentally.*

$$[\boldsymbol{K}]\{\boldsymbol{D}\} + (\boldsymbol{\alpha}[\boldsymbol{K}] + \boldsymbol{\beta}[\boldsymbol{M}])\{\dot{\boldsymbol{D}}\} + [\boldsymbol{M}]\{\ddot{\boldsymbol{D}}\} = \{\boldsymbol{F}(\boldsymbol{t})\} \qquad (4.0)$$

The equation can be solved numerically using the modal superposition method or the direct time integration method.