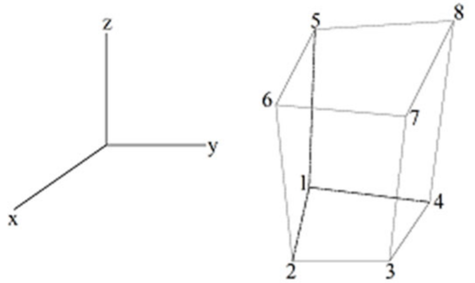
3.2.2. Hexahedral elements:

The hexahedral element is an eight-node, six-face element, with the nodes numbered counterclockwise



As there are three degrees of freedom (3DOF) at each node, there is a total of twenty-four DOF for the element. We can work in the local coordinate system of the element and follow the steps already used for one-dimensional and two-dimensional elements.

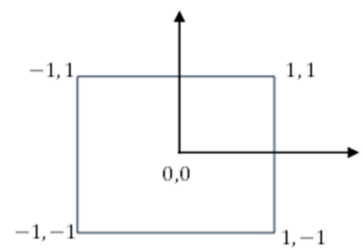
Each component of the displacement field is approximated as follows:

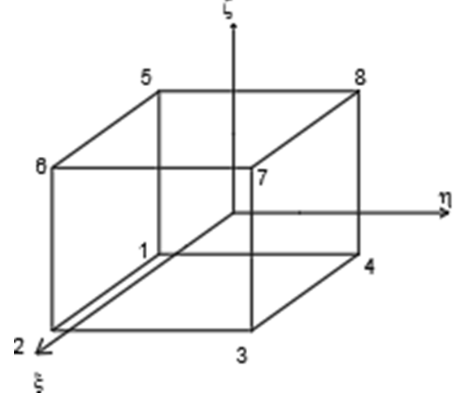
$$u = a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xz + a_6yz + a_7xyz$$

But it is useful to define a reference element in a natural coordinate system with its origin at the center of the element, as this facilitates the construction of shape functions and the evaluation of matrix integration.

For the linear one-dimensional element designed for the study of trusses (tensile and compressive bar system), the reference element and the associated natural coordinate system are given by the following figure:

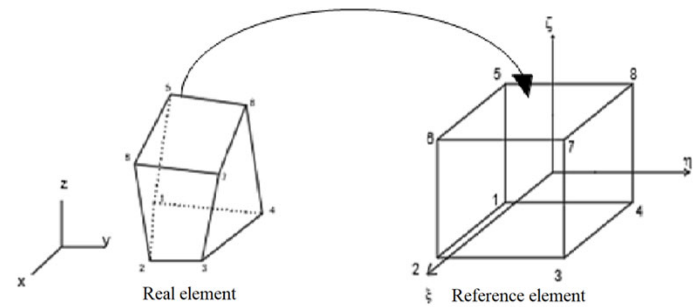
For the quadrilateral plane element designed for the study of plane elasticity problems, for example, the reference element and the associated natural coordinate system are given by the following figure:





1(-1,-1,-1)
2(-1,1,-1)
3(1,1,-1)
4(1,-1,-1)
5(-1,-1,1)
6(-1,1,1)
7(1,1,1)
8(1,-1,1)

The transition from the reference element to the real element is done through geometric transformations.



We often use the same shape functions found in the natural system as interpolation functions for coordinates (iso parametric elements):

$$x = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta) \times \xi_i$$

$$y = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta) \times \eta_i$$

$$z = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta) \times \zeta_i$$

3.2.2.1. Approximation of the displacement field:

In the natural coordinate system:

$$\boldsymbol{u = a_0 + a_1\xi + a_2\eta + a_3\zeta + a_4\xi\eta + a_5\xi\zeta + a_6\eta\zeta + a_7\xi\eta\zeta}$$

$$\boldsymbol{u = \lfloor 1 \;\; \xi \;\; \eta \;\; \zeta \;\; \xi\eta \;\; \xi\zeta \;\; \eta\zeta \;\; \xi\eta\zeta \rfloor \{a\}}$$

The boundary conditions give:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix} = 0.125 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$\boldsymbol{u} = \lfloor 1 \quad \xi \quad \eta \quad \zeta \quad \xi\eta \quad \xi\zeta \quad \eta\zeta \quad \xi\eta\zeta \rfloor \times 0.125 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$\boldsymbol{u}(\xi,\eta,\zeta) = \sum_{i=1}^{8} N_i\,(\xi,\eta,\zeta)u_i$$

Where:

$$N_1 = 0.125(1-\xi)(1-\eta)(1-\zeta)$$
$$N_2 = 0.125(1+\xi)(1-\eta)(1-\zeta)$$
$$N_3 = 0.125(1+\xi)(1+\eta)(1-\zeta)$$
$$N_4 = 0.125(1-\xi)(1+\eta)(1-\zeta)$$
$$N_5 = 0.125(1-\xi)(1-\eta)(1+\zeta)$$
$$N_6 = 0.125(1+\xi)(1-\eta)(1+\zeta)$$
$$N_7 = 0.125(1+\xi)(1+\eta)(1+\zeta)$$
$$N_8 = 0.125(1-\xi)(1+\eta)(1+\zeta)$$

In a hexahedral element, the displacement vector U is a function of the coordinates x, y and z as previously, it is interpolated using shape functions:

$$\boldsymbol{U} = \begin{Bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{Bmatrix} = [N]\{d\}$$

Where:

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 \end{bmatrix}$$

$$\{d\} = \lfloor u_1 \;\; v_1 \;\; w_1 \;\; u_2 \;\; v_2 \;\; w_2 \;\; u_3 \;\; v_3 \;\; w_3 \;\; u_4 \;\; v_4 \;\; w_4 \;\; u_1 \;\; v_1 \;\; w_1 \;\; u_2 \;\; v_2 \;\; w_2 \;\; u_3 \;\; v_3 \;\; w_3 \;\; u_4 \;\; v_4 \;\; w_4 \rfloor^T$$

To condense the writing, let's express the previous system in the form:

$$\boldsymbol{U} = [\boldsymbol{N}]\{\boldsymbol{d}\}$$

Where:

$$[\boldsymbol{N}] = \lfloor \boldsymbol{N_1} \quad \boldsymbol{N_2} \quad \boldsymbol{N_3} \quad \boldsymbol{N_4} \quad \boldsymbol{N_5} \quad \boldsymbol{N_6} \quad \boldsymbol{N_7} \quad \boldsymbol{N_8} \rfloor$$

$$[\boldsymbol{d}] = \lfloor \boldsymbol{d_1} \quad \boldsymbol{d_2} \quad \boldsymbol{d_3} \quad \boldsymbol{d_4} \quad \boldsymbol{d_5} \quad \boldsymbol{d_6} \quad \boldsymbol{d_7} \quad \boldsymbol{d_8} \rfloor^{\boldsymbol{T}}$$

With:

$$[\boldsymbol{N_i}] = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix}; and\ \{\boldsymbol{d}_i\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} = \begin{Bmatrix} d_{3(i-1)+1} \\ d_{3(i-1)+2} \\ d_{3(i-1)+3} \end{Bmatrix}; i = 1, 2, 3, \ldots., 8$$

3.2.2.2. Deformation field:

The shape functions are defined in natural coordinates, ξ, η et ζ, while differentiation is performed in the local coordinates x, y, z:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\varepsilon = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \lfloor \boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3 \quad \boldsymbol{N}_4 \quad \boldsymbol{N}_5 \quad \boldsymbol{N}_6 \quad \boldsymbol{N}_7 \quad \boldsymbol{N}_8 \rfloor \begin{Bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \\ \boldsymbol{d}_4 \\ \boldsymbol{d}_5 \\ \boldsymbol{d}_6 \\ \boldsymbol{d}_7 \\ \boldsymbol{d}_8 \end{Bmatrix}$$

$$\varepsilon = \lfloor \boldsymbol{B}_1 \quad \boldsymbol{B}_2 \quad \boldsymbol{B}_3 \quad \boldsymbol{B}_4 \quad \boldsymbol{B}_5 \quad \boldsymbol{B}_6 \quad \boldsymbol{B}_7 \quad \boldsymbol{B}_8 \rfloor \begin{Bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \boldsymbol{d}_3 \\ \boldsymbol{d}_4 \\ \boldsymbol{d}_5 \\ \boldsymbol{d}_6 \\ \boldsymbol{d}_7 \\ \boldsymbol{d}_8 \end{Bmatrix}$$

Where:

The shape functions are defined in natural coordinates, ξ η, ξ, while differentiation is performed in the local coordinates x, y, z

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \xi}$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial N_i}{\partial \zeta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial \zeta}$$

Or in matrix form:

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \boldsymbol{J} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix}; \text{ J is the Jacobian matrix}$$

Since:

$x = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) \times \xi_i$; $y = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) \times \eta_i$; $z = \sum_{i=1}^{8} N_i(\xi, \eta, \zeta) \times \zeta_i$

$$J = \begin{bmatrix} \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \zeta} & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \zeta} & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \zeta} \end{bmatrix}$$

Or in expanded form

$$J = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \frac{\partial N_5}{\partial \xi} & \frac{\partial N_6}{\partial \xi} & \frac{\partial N_7}{\partial \xi} & \frac{\partial N_8}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & \frac{\partial N_5}{\partial \eta} & \frac{\partial N_6}{\partial \eta} & \frac{\partial N_7}{\partial \eta} & \frac{\partial N_8}{\partial \eta} \\ \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \frac{\partial N_3}{\partial \zeta} & \frac{\partial N_4}{\partial \zeta} & \frac{\partial N_5}{\partial \zeta} & \frac{\partial N_6}{\partial \zeta} & \frac{\partial N_7}{\partial \zeta} & \frac{\partial N_8}{\partial \zeta} \end{bmatrix}$$

We have

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \boldsymbol{J} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix}, \qquad So, \qquad \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \boldsymbol{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix}$$

This is used to calculate the strain matrix, $\boldsymbol{B}$, by replacing all derivatives of the shape functions with respect to x, y, z with those with respect to ξ, η, ζ

3.2.2.3. Stresses field:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = [C]\varepsilon = \frac{\lambda}{v} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

3.2.2.4. Stiffness matrix:

$$[\boldsymbol{k}] = \int_V [\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}]\, d\boldsymbol{v} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}] \boldsymbol{det\,J}\, \boldsymbol{d\xi\, d\eta\, d\zeta}$$

Since evaluating the integrations of the previous equation can be tedious, the integrals are therefore performed using a numerical integration scheme.

The Gauss integration scheme is often used to perform the integral. For three-dimensional integrations, the Gauss integration takes the following form:

$$\boldsymbol{I} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{f}(\xi,\eta,\zeta)\, \boldsymbol{d\xi\, d\eta\, d\zeta} = \sum_{i=1}^{n}\sum_{i=1}^{m}\sum_{i=1}^{l} \boldsymbol{w_i w_j\, w_k\, f}(\xi_i, \eta_j, \zeta_k)$$