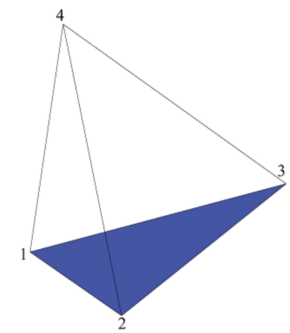
3.2 Finite elements in three dimensions

Three-dimensional finite elements (solid elements) are used to solve problems described by partial differential equations with three variables. In solid mechanics problems, there are a total of six possible stress components: three normal stresses and three shear stresses, which must be considered. Generally, a 3D solid element can be tetrahedral or hexahedral in shape, with flat or curved surfaces. Each node of the element will have three translational degrees of freedom (3DOF). Thus, the element can deform in all three spatial directions

3.2.1 Tetrahedral element (4 Nodes; 4 faces):



3.2.1.1. Displacements field:

$$\{\boldsymbol{d}\} = \lfloor \boldsymbol{u_1} \quad \boldsymbol{v_1} \quad \boldsymbol{w_1} \quad \boldsymbol{u_2} \quad \boldsymbol{v_2} \quad \boldsymbol{w_2} \quad \boldsymbol{u_3} \quad \boldsymbol{v_3} \quad \boldsymbol{w_3} \quad \boldsymbol{u_4} \quad \boldsymbol{v_4} \quad \boldsymbol{w_4} \rfloor^{\boldsymbol{T}}$$

$$u = N_1U_1 + N_2U_2 + N_3U_3 + N_4U_4 \ ; \quad likewise\ \ v \text{ and } w$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \{d\}$$

The shape functions N_i can be determined by approximating each component with a 4-term polynomial:

$$u = a_0 + a_1x + a_2y + a_3z = \lfloor 1 \quad x \quad y \quad z \rfloor \{a\}$$

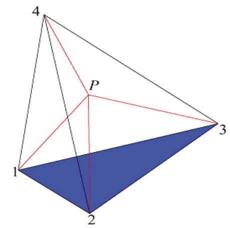
and using the boundary conditions that lead to the system:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \{a\}$$

Solving this system using Cramer's method gives the ai in terms of the ui , and by substituting into the expression for u; we can determine the Ni.

However, we prefer to use the volume coordinates L:

$$L_1 = \frac{V_{p234}}{V} \quad , \quad L_2 = \frac{V_{p134}}{V} \quad , \quad L_3 = \frac{V_{p124}}{V} \quad , \quad L_4 = \frac{V_{p123}}{V}$$



Where V is the volume of the element and V_{pijk} *is the volume of the tetrahedron whose vertices are an interior point p and the vertices* i, j, k *of the element.*

We deduce that:

$$L_1 + L_2 + L_3 + L_4 = 1\,; \quad V_{p234} + V_{p134} + V_{p124} + V_{p123} = V$$

If p becomes one of the vertices i of the tetrahedron we will have $L_i = 1$ which allows one to write:

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4$$
$$y = L_1 y_1 + L_2 y_2 + L_3 y_3 + L_4 y_4$$
$$z = L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4$$

From which we can form the following system:

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix}$$

Let's use the cofactor method to determine the inverse matrix:

For a matrix $[A]$: $\quad [A]^{-1} = \frac{1}{det[A]} [com[A]]^T$

The terms of : $\left[com \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \right]$ *are:*

$$c_{11} = det \begin{bmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix};\; c_{12} = det \begin{bmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix};\; c_{13} = det \begin{bmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{14} = det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{21} = det \begin{bmatrix} 1 & 1 & 1 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix};\; c_{22} = det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix};\; c_{23} = det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{24} = det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{31} = det \begin{bmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \end{bmatrix};\; c_{32} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ z_1 & z_3 & z_4 \end{bmatrix};\; c_{33} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{34} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{41} = det \begin{bmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{bmatrix};\; c_{42} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{bmatrix};\; c_{43} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{bmatrix}$$

$$c_{44} = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Noting that:

$$det\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} = 6V$$

Then:

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

with:

$$a_i = c_{1i}{}^T \ ; \ b_i = c_{2i}{}^T \ ; \ c_i = c_{3i}{}^T \ ; \ d_i = c_{4i}{}^T$$

$$N_i = L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z)$$

3.2.1.2. Deformations field:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{Bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\varepsilon = [B]\{d\} = \begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ 0 & d_1 & c_1 & 0 & d_2 & c_2 & 0 & d_3 & c_3 & 0 & d_4 & c_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \end{bmatrix} \{d\}$$

3.2.1.3. Stresses field:

$$\sigma = [C]\varepsilon$$

$$\sigma == \frac{\boldsymbol{E}}{(\boldsymbol{1-2\nu})(\boldsymbol{1+\nu})} \begin{bmatrix} \boldsymbol{1-\nu} & \boldsymbol{\nu} & \boldsymbol{\nu} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\nu} & \boldsymbol{1-\nu} & \boldsymbol{\nu} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\nu} & \boldsymbol{\nu} & \boldsymbol{1-\nu} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{1-2\nu})/\boldsymbol{2} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{1-2\nu})/\boldsymbol{2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{1-2\nu})/\boldsymbol{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

3.1.2.4. Stiffness matrix of the element:

$$[\boldsymbol{k}] = \int_{\boldsymbol{V}} [\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}]\boldsymbol{dv}$$

As $[\boldsymbol{B}]$*and* $[\boldsymbol{C}]$ *are constants, then:*

$$[\boldsymbol{k}] = \boldsymbol{V}[\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}]$$