

3.1.3 Formulation of the plate element for the study of plane elasticity problems:

Types of plane elasticity problems

Plane stresses: the stresses are in a plane:

If the plane is x-y, the stresses are a function of (x,y) :

$$\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0 \text{ or in engineering notations: } \sigma_z = \tau_{yz} = \tau_{xz} = 0$$

Which gives in deformations:

$$\varepsilon_{yz} = \varepsilon_{xz} = 0, \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \text{ or in engineering notations: } \varepsilon_{yz} = \varepsilon_{xz} = 0,$$

$$\text{and } \sigma_z = -\frac{\nu}{E}(\sigma_x + \sigma_z)$$

So, the displacement field is: $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}, w = 0$

Plane deformations: the deformations are in a plane:

If the plane is x-y, the deformations are a function of (x,y) :

$$\varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{xz} = 0, \text{ or in engineering notations: } \varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

Which gives in stresses:

$$\sigma_{yz} = \sigma_{xz} = 0, \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \text{ or in engineering notations: } \varepsilon_{yz} = \varepsilon_{xz} = 0,$$

$$\text{and } \sigma_z = -\frac{\nu}{E}(\sigma_x + \sigma_z)$$

So, the displacement field is: $\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}, w = 0$

The stresses are :

$$\sigma_{yz} = \sigma_{xz} = 0, \sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{zz}) \text{ or in engineering notations: } \tau_{yz} = \tau_{xz} = 0,$$

et $\sigma_z = \lambda(\varepsilon_x + \varepsilon_z)$ is a function of (x,y) where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

3.1.3.1. Formulation of the triangular element

3.1.3.1.1. Basic equations:

✓ Displacements field

$$U = [N]\{d\}; U = \begin{Bmatrix} u \\ v \end{Bmatrix}; \{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}; [N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

✓ deformations field:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

$$\varepsilon = [B]\{d\} \rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

✓ **Stresses field:**

In the 3D case we have:

$$\{\sigma\} = [C]\{\varepsilon\} \text{ ou } \{\varepsilon\} = [C]^{-1}\{\sigma\} \text{ où : } \{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \text{ and } \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

$$[C] = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

In the plane stresses case we have

$$\sigma_{zz} = 0 \rightarrow \frac{\lambda}{v}(v(+\varepsilon_{yy}) + (1-v)\varepsilon_{zz}) = 0 \rightarrow \varepsilon_{zz} = \frac{-v}{1-v}(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\sigma_{xx} = \frac{\lambda}{v}((1-v)\varepsilon_{xx} + v\varepsilon_{yy} + v\frac{-v}{1-v}(\varepsilon_{xx} + \varepsilon_{yy}))$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{\nu E}{1-\nu^2}\varepsilon_{yy}; \quad \sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} + \frac{E}{1-\nu^2}\varepsilon_{yy}; \quad \sigma_{xy} = \frac{E}{2(1+\nu)}2\varepsilon_{xy}$$

$$\{\sigma\} = [C]\{\varepsilon\} \rightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

In the plane deformations case we have:

$$\{\sigma\} = [C]\{\varepsilon\} \rightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

3.1.3.1.2. **Formulation of the element:**

$$\pi = U + P$$

$$U = \int_V \frac{1}{2} \varepsilon^T \sigma dV = \frac{1}{2} \int_V ([B]\{d\})^T [C][B]\{d\} dV = \frac{1}{2} \{d\}^T \left(\int_V [B]^T [C][B] dV \right) \{d\} = \frac{1}{2} \{d\}^T [k] \{d\}$$

$$P = - \left(\{d\}^T \{f_n\} + \int_{\Gamma} U^T \{T\} ds + \int_V U^T \{X\} dv \right) = - \left(\{d\}^T \{f_n\} + \int_{\Gamma} U^T \{T\} ds + \int_V U^T \{X\} dv \right)$$

$$P = - \left(\{d\}^T \{f_n\} + \int_{\Gamma} ([N]\{d\})^T \{T\} ds + \int_V ([N]\{d\})^T \{X\} dv \right)$$

$$P = - \{d\}^T \left(\{f_n\} + \int_{\Gamma} [N]^T \{T\} ds + \int_V [N]^T \{X\} dv \right) = - \{d\}^T \{f\}$$

$$\pi = \frac{1}{2} \{d\}^T \left(\int_V [B]^T [C][B] dV \right) \{d\} - \{d\}^T \{f\}$$

The minimization of the potential energy which is expressed by $\frac{\partial \pi}{\partial d_i} = 0$ gives :

$$\left(\int_V [B]^T [C] [B] dv \right) \{d\} - \{f\} = 0$$

$$\{f\} = [k]\{d\} \quad \text{avec} \quad [k] = \int_V [B]^T [C] [B] dv$$

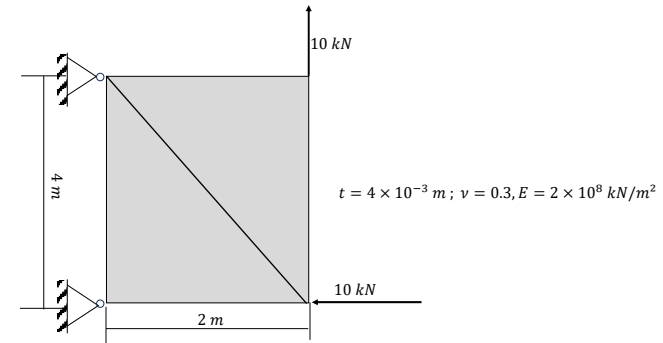
As $[B]$ and $[C]$ are constants :

$$[k] = \int_V [B]^T [C] [B] dv = [B]^T [C] [B] \int_V dv$$

For a plate of constant thickness t : $\int_V dv = \Delta t$

$$[k] = [B]^T [C] [B] \Delta t$$

3.1.3.1.3. Example:



Element 1 :

$E = 2 \times 10^8 \text{ kN/m}^2$; $\nu = 0.3$; $(x_1, y_1) = (0, 0)$; $(x_2, y_2) = (2, 0)$; $(x_3, y_3) = (0, 4)$; $t = 0.004$; $\Delta = 4$

$$[B]^1 = \begin{bmatrix} -0.50 & 0.00 & 0.50 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.25 & 0.00 & 0.00 & 0.00 & 0.25 \\ -0.25 & -0.50 & 0.00 & 0.50 & 0.25 & 0.00 \end{bmatrix}$$

$$[C]^1 = \begin{bmatrix} 219780219.78 & 65934065.93 & 0.00 \\ 65934065.93 & 219780219.78 & 0.00 \\ 0.00 & 0.00 & 76923076.92 \end{bmatrix}$$

$$[k]^1 = \begin{bmatrix} 956043.96 & 285714.29 & -879120.88 & -153846.15 & -76923.08 & -131868.13 \\ 285714.29 & 527472.53 & -131868.13 & -307692.31 & -153846.15 & -219780.22 \\ -879120.88 & -131868.13 & 879120.88 & 0.000000 & 0.000000 & 131868.13 \\ -153846.15 & -307692.31 & 0.000000 & 307692.31 & 153846.15 & 0.000000 \\ -76923.08 & -153846.15 & 0.000000 & -153846.15 & 76923.08 & 0.000000 \\ -131868.13 & -219780.22 & 131868.13 & 0.000000 & 0.000000 & 219780.22 \end{bmatrix}$$

Element 2 :

$$E = 2 \times 10^8 \frac{kN}{m^2}; \nu = 0.3; (x_1, y_1) = (2, 0); (x_2, y_2) = (2, 4); (x_3, y_3) = (0, 4); t = 0.004; \Delta = 4$$

$$[B]^2 = \begin{bmatrix} 0.00 & 0.00 & 0.50 & 0.00 & -0.50 & 0.00 \\ 0.00 & -0.25 & 0.00 & 0.25 & 0.00 & 0.00 \\ -0.25 & 0.00 & 0.25 & 0.50 & 0.00 & -0.50 \end{bmatrix}$$

$$[C]^2 = \begin{bmatrix} 219780219.78 & 65934065.93 & 0.00 \\ 65934065.93 & 219780219.78 & 0.00 \\ 0.00 & 0.00 & 76923076.92 \end{bmatrix}$$

$$[k]^2 = \begin{bmatrix} 76923.08 & 0.0000000 & -76923.08 & -153846.15 & 0.0000000 & 153846.15 \\ 0.0000000 & 219780.22 & -131868.13 & -219780.22 & 131868.13 & 0.0000000 \\ -76923.08 & -131868.13 & 956043.96 & 285714.29 & -879120.88 & -153846.15 \\ -153846.15 & -219780.22 & 285714.29 & 527472.53 & -131868.13 & -307692.31 \\ 0.0000000 & 131868.13 & -879120.88 & -131868.13 & 879120.88 & 0.0000000 \\ -153846.15 & 0.0000000 & -153846.15 & -307692.31 & 0.0000000 & 307692.31 \end{bmatrix}$$

Boundary conditions:

$$D_1 = D_2 = D_5 = D_6 = 0$$

$$[k]^* = \begin{bmatrix} 956043.96 & 0.0000000 & -76923.08 & -153846.15 \\ 0.0000000 & 527472.53 & -131868.13 & -219780.22 \\ -76923.08 & -131868.13 & 956043.96 & 285714.29 \\ -153846.15 & -219780.22 & 285714.29 & 527472.53 \end{bmatrix}$$

$$\{F\}^* = \begin{pmatrix} -10 \\ 0 \\ 0 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} -10 \\ 0 \\ 0 \\ 10 \end{pmatrix} = \begin{bmatrix} 956043.96 & 0.0000000 & -76923.08 & -153846.15 \\ 0.0000000 & 527472.53 & -131868.13 & -219780.22 \\ -76923.08 & -131868.13 & 956043.96 & 285714.29 \\ -153846.15 & -219780.22 & 285714.29 & 527472.53 \end{bmatrix} \begin{pmatrix} D_3 \\ D_4 \\ D_7 \\ D_8 \end{pmatrix}$$

$$\begin{pmatrix} D_3 \\ D_4 \\ D_7 \\ D_8 \end{pmatrix} = \begin{pmatrix} -7.143 \\ 8.315 \\ -6.570 \\ 23.898 \end{pmatrix} \times 10^{-6}$$

Stresses in the elements:

$$\text{Element 1 : } \{d\}^1 = \begin{Bmatrix} 0 \\ 0 \\ -7.143 \\ 8.315 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma^1 = [C]^1 [B]^1 \{d\}^1$$

$$\sigma^1 = \begin{bmatrix} 219780219.78 & 65934065.93 & 0.00 \\ 65934065.93 & 219780219.78 & 0.00 \\ 0.00 & 0.00 & 76923076.92 \end{bmatrix} \begin{bmatrix} -0.50 & 0.00 & 0.50 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.25 & 0.00 & 0.00 & 0.00 & 0.25 \\ -0.25 & -0.50 & 0.00 & 0.50 & 0.25 & 0.00 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -7.143 \\ 8.315 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-6}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^1 = \begin{Bmatrix} -784.94 \\ -235.48 \\ 319.81 \end{Bmatrix}$$

$$\text{Element 2 : } \{d\}^1 = \begin{Bmatrix} -7.143 \\ 8.315 \\ -6.570 \\ 23.898 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-6}$$

$$\sigma^2 = [C]^2 [B]^2 \{d\}^2$$

$$\sigma^2 = \begin{bmatrix} 219780219.78 & 65934065.93 & 0.00 \\ 65934065.93 & 219780219.78 & 0.00 \\ 0.00 & 0.00 & 76923076.92 \end{bmatrix} \begin{bmatrix} 0.00 & 0.00 & 0.50 & 0.00 & -0.50 & 0.00 \\ 0.00 & -0.25 & 0.00 & 0.25 & 0.00 & 0.00 \\ -0.25 & 0.00 & 0.25 & 0.50 & 0.00 & -0.50 \end{bmatrix} \begin{Bmatrix} -7.143 \\ 8.315 \\ -6.570 \\ 23.898 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-6}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^2 = \begin{Bmatrix} -465.09 \\ 639.63 \\ 930.18 \end{Bmatrix}$$

3.1.3.2 Formulation of the rectangular element

$$U = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_2 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix}, \quad \text{avec} \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{ab} \begin{bmatrix} -b+y & 0 & b-y & 0 & y & 0 & -y & 0 \\ 0 & -a+x & 0 & -x & 0 & x & 0 & a-x \\ -a+x & -b+y & -x & b-y & x & y & a-x & -y \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix}$$

$$\{\sigma\} = [C]\{\varepsilon\}$$

$$\text{In plane stresses} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}; \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

In plane stresses:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}; \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

For a plate with constant thickness t:

$$[k] = t \int_0^a \int_0^b [B]^T [C] [B] dx dy$$