Chapter 3 : Finite element in two and three dimensions

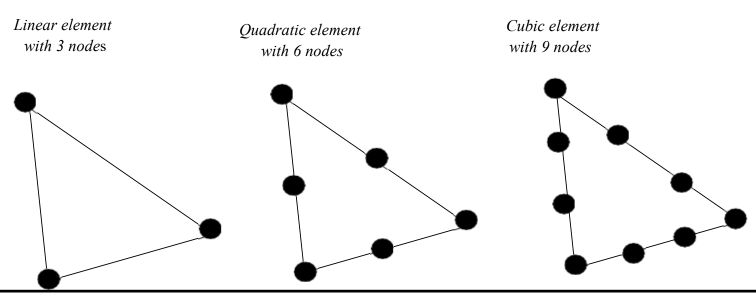
3.1. Finite element in two dimensions

Two-dimensional finite elements are used to solve problems described by partial differential equations with two variables. Many practical engineering problems can be conceptually defined in two dimensions and can be described by two-dimensional partial differential equations, especially when the primary unknowns vary mainly in two directions while remaining constant in the third direction.

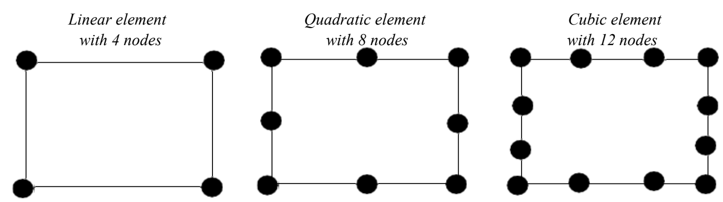
Plane elasticity, axisymmetric problems, and two-dimensional fluid flow are examples of problems that can be studied using two-dimensional finite elements."

1. 3.1.1. *Commonly used elements and shape functions:*

Triangular element with 3, 6, or 9 nodes

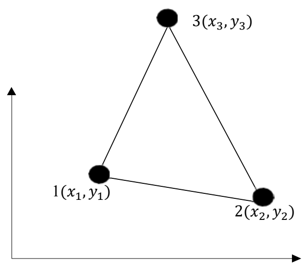


Quadrilateral elements with 4, 8, or 12 nodes (especially the rectangular element)



3.1.2. Shape functions:

3.1.2.1. Triangular element:



Each component will be approximated separately:

$$u(x,y) = a_0 + a_1 x + a_2 y = \lfloor 1 \quad x \quad y \rfloor \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}$$

Boundary conditions are:

Node1 : $(x,y) = (x_1, y_1) \rightarrow u(x,y) = u_1 = a_0 + a_1 x_1 + a_2 y_1$

Node2 : $(x,y) = (x_2, y_2) \rightarrow u(x,y) = u_2 = a_0 + a_1 x_2 + a_2 y_2$

Node 3 : $(x,y) = (x_3, y_2) \rightarrow u(x,y) = u_3 = a_0 + a_1 x_3 + a_2 y_3$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}, In\ condensed\ form\ \{d\} = [\bar{P}]\{a\}$$

To determine $\{a\}$ *we invert* $[\bar{P}]$ *, but we prefer to solve the system using Cramer's method.*

It should be noted that the determinant of $[\bar{P}]$ *is not zero since it is equal to twice the area of triangle 1 2 3. Let's denote its value by* 2Δ*, Thus, the solution will be*

$$a_0 = \frac{\begin{bmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}} = \frac{1}{2\Delta}\left((x_2 y_3 - y_2 x_3)u_1 - x_1(u_2 y_3 - y_2 u_3) + y_1(u_2 x_3 - x_2 u_3)\right)$$

$$a_0 = \frac{1}{2\Delta}((x_2 y_3 - y_2 x_3)u_1 + (y_1 x_3 - x_1 y_3)u_2 + (x_1 y_2 - y_1 x_2)u_3)$$

$$a_1 = \frac{\begin{bmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}} = \frac{1}{2\Delta}((y_3 u_2 - u_3) - u_1(y_3 - y_2) + y_1(u_3 - u_2))$$

$$a_1 = \frac{1}{2\Delta}(((y_2 - y_3)u_1 + (y_3 - y_1)u_2 + (y_1 - y_2)u_3$$

$$a_2 = \frac{\begin{bmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{bmatrix}}{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}} = \frac{1}{2\Delta}((x_2 u_3 - u_2 x_3) - x_1(u_3 - u_2) + u_1(x_3 - x_2))$$

$$a_2 = \frac{1}{2\Delta}((x_3 - x_2)u_1 + (x_1 - x_3)u_2 + (x_2 - x_1)u_3)$$

Hence :

$$\{a\} = [\bar{P}]^{-1}\{u\} \rightarrow \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} x_2y_3 - y_2x_3 & y_1x_3 - x_1y_3 & x_1y_2 - y_1x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [P]\{a\} = \lfloor 1 \quad x \quad y \rfloor \times \frac{1}{2\Delta}\begin{bmatrix} x_2y_3 - y_2x_3 & y_1x_3 - x_1y_3 & x_1y_2 - y_1x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

Where :

$$N_1 = \frac{1}{2\Delta}[(x_2y_3 - y_2x_3) + x(y_2 - y_3) + y(x_3 - x_2)]$$

$$N_2 = \frac{1}{2\Delta}[(y_1x_3 - x_1y_3) + x(y_3 - y_1) + y(x_1 - x_3)]$$

$$N_3 = \frac{1}{2\Delta}[(x_1y_2 - y_1x_2) + x(y_1 - y_2) + y(x_2 - x_1)]$$

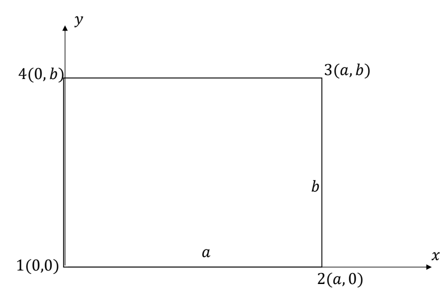
A similar reasoning for the component v gives :

$$v = N_1v_1 + N_2v_2 + N_3v_3$$

The complete field is written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}, \quad where \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

3.1.2.2. Rectangular element



$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy = \lfloor 1 \quad x \quad y \quad xy \rfloor \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Boundary conditions are:

Node 1 : $(x,y) = (0,0) \rightarrow u(x,y) = u_1 = a_0$

Node 2 : $(x,y) = (a,0) \rightarrow u(x,y) = u_2 = a_0 + a_1 a$

Node 3 : $(x,y) = (a,b) \rightarrow u(x,y) = u_3 = a_0 + a_1 a + a_2 b + a_3 ab$

Node 4 : $(x,y) = (0,b) \rightarrow u(x,y) = u_4 = a_0 + a_2 b$

$$a_0 = u_1, \qquad a_1 = \frac{1}{a}(u_2 - u_1), \qquad a_2 = \frac{1}{b}(u_4 - u_1), \qquad a_3 = \frac{1}{ab}(u_3 + u_1 - u_2 - u_4$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 & 0 \\ -b & b & 0 & 0 \\ -a & 0 & 0 & a \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u(x,y) = \lfloor 1 \quad x \quad y \quad xy \rfloor \times \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 & 0 \\ -b & b & 0 & 0 \\ -a & 0 & 0 & a \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

Where :

$$N_1 = \frac{1}{ab}(ab - bx - ay + xy) = 1 - \frac{x}{a} - \frac{y}{b} + \frac{xy}{ab}$$

$$N_2 = \frac{1}{ab}(bx - xy) = \frac{x}{a} - \frac{xy}{ab}$$

$$N_3 = \frac{xy}{ab}$$

$$N_4 = \frac{1}{ab}(ay - xy) = \frac{y}{b} - \frac{xy}{ab}$$

A similar reasoning for the component v gives:

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

The full field is written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix}, \qquad avec \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$