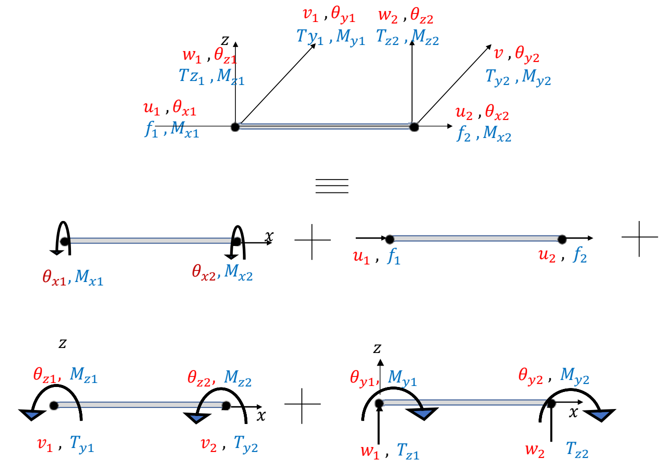
2_6 : Bar element of tension-compression and bending in two orthogonal planes and torsion (3D Frame element)

2.6.1 : Definition, Notations and Sign Conventions

This element is the combination of a tension-compression bar element, a torsion bar element and a bending bar element in two mutually orthogonal planes

The displacement field is:

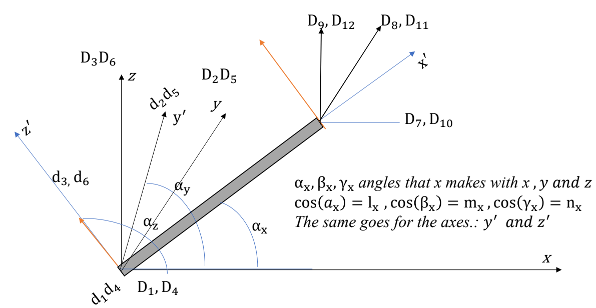
$$\begin{Bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}$$



The stiffness matrix of this element is therefore:

$$[k] = \begin{bmatrix}
A & 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12I_z}{L^2} & 0 & 0 & 0 & \frac{6I_z}{L} & 0 & \frac{-12I_z}{L^2} & 0 & 0 & 0 & \frac{6I_z}{L} \\
0 & 0 & \frac{12I_y}{L^2} & 0 & \frac{-6I_y}{L} & 0 & 0 & 0 & \frac{-12I_y}{L^2} & 0 & \frac{-6I_y}{L} & 0 \\
0 & 0 & 0 & \frac{J}{2(1+v)} & 0 & 0 & 0 & 0 & 0 & \frac{-J}{2(1+v)} & 0 & 0 \\
0 & 0 & \frac{-6I_y}{L} & 0 & 4I_y & 0 & 0 & 0 & \frac{6I_y}{L} & 0 & 2I_y & 0 \\
0 & \frac{6I_z}{L} & 0 & 0 & 0 & 4I_z & 0 & \frac{-6I_z}{L} & 0 & 0 & 0 & 2I_z \\
-A & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12I_z}{L^2} & 0 & 0 & 0 & \frac{-6I_z}{L} & 0 & \frac{12I_z}{L^2} & 0 & 0 & 0 & \frac{-6I_z}{L} \\
0 & 0 & \frac{-12I_y}{L^2} & 0 & \frac{6I_y}{L} & 0 & 0 & 0 & \frac{12I_y}{L^2} & 0 & \frac{6I_y}{L} & 0 \\
0 & 0 & 0 & \frac{-J}{2(1+v)} & 0 & 0 & 0 & 0 & 0 & \frac{J}{2(1+v)} & 0 & 0 \\
0 & 0 & \frac{-6I_y}{L} & 0 & 2I_y & 0 & 0 & 0 & \frac{6I_y}{L} & 0 & 4I_y & 0 \\
0 & \frac{6I_z}{L} & 0 & 0 & 0 & 2I_z & 0 & \frac{-6I_z}{L} & 0 & 0 & 0 & 4I_z
\end{bmatrix}$$

The transformation matrix is obtained as follows:



The projection of the first three degrees of freedom (the translations at node 1) onto the x', y' and z' axes gives:

$$d_1 = \mathrm{l_x} \mathrm{D}_1 + \mathrm{m_x}\, \mathrm{D}_2 + \mathrm{n_x} \mathrm{D}_3$$
$$d_2 = \mathrm{l_y} \mathrm{D}_1 + \mathrm{m_y} \mathrm{D}_2 + \mathrm{n_y}\, \mathrm{D}_3$$
$$d_3 = \mathrm{l_z} \mathrm{D}_1 + \mathrm{m_z}\, \mathrm{D}_2 + \mathrm{n_z}\, \mathrm{D}_3$$

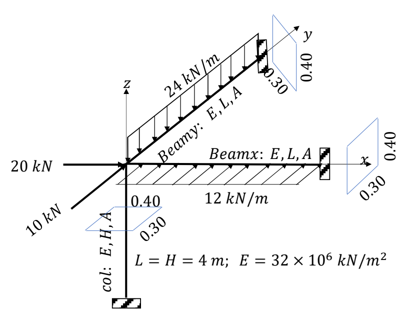
$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = [T1] \begin{Bmatrix} \mathrm{D}_1 \\ \mathrm{D}_2 \\ \mathrm{D}_3 \end{Bmatrix} \quad where\ [\boldsymbol{T1}] = \begin{bmatrix} \boldsymbol{l_x} & \boldsymbol{m_x} & \boldsymbol{n_x} \\ \boldsymbol{l_y} & \boldsymbol{m_y} & \boldsymbol{n_y} \\ \boldsymbol{l_z} & \boldsymbol{m_z} & \boldsymbol{n_z} \end{bmatrix}$$

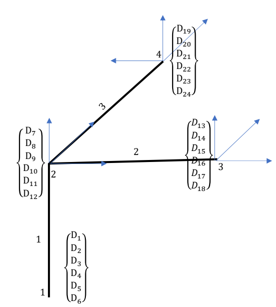
A similar work for rotations at node 1 as well as for translations and rotations at node 2 gives:

$$\begin{Bmatrix} d_4 \\ d_5 \\ d_6 \end{Bmatrix} = [T1] \begin{Bmatrix} \mathrm{D}_4 \\ \mathrm{D}_5 \\ \mathrm{D}_6 \end{Bmatrix}; \quad \begin{Bmatrix} d_7 \\ d_8 \\ d_9 \end{Bmatrix} = [T1] \begin{Bmatrix} \mathrm{D}_7 \\ \mathrm{D}_8 \\ \mathrm{D}_9 \end{Bmatrix}; \quad \begin{Bmatrix} d_{10} \\ d_{11} \\ d_{12} \end{Bmatrix} = [T1] \begin{Bmatrix} \mathrm{D}_{10} \\ \mathrm{D}_{11} \\ \mathrm{D}_{12} \end{Bmatrix}$$

$$[T] = \begin{bmatrix} [T1] & [Z3] & [Z3] & [Z3] \\ [Z3] & [T1] & [Z3] & [Z3] \\ [Z3] & [Z3] & [T1] & [Z3] \\ [Z3] & [Z3] & [Z3] & [T1] \end{bmatrix} \quad avec\ [Z3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example:





Continuity of displacements:

$$\{D\}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \end{Bmatrix}; \{D\}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \end{Bmatrix}^2 = \begin{Bmatrix} D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{14} \\ D_{15} \\ D_{16} \\ D_{17} \\ D_{18} \end{Bmatrix}; \{D\}^3 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \end{Bmatrix}^3 = \begin{Bmatrix} D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{19} \\ D_{20} \\ D_{21} \\ D_{22} \\ D_{23} \\ D_{24} \end{Bmatrix}$$

Boundary conditions:

$D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = 0$

$D_{13} = D_{14} = D_{15} = D_{16} = D_{17} = D_{18} = 0$

$D_{19} = D_{20} = D_{21} = D_{22} = D_{23} = D_{24} = 0$

To simplify the calculations, let's introduce the boundary conditions at the element level

Element 1: $L = 4, A = 0.12, I_{y'} = 0.0009, I_{z'} = 0.0016, \quad J = 0.001994$

$$\alpha_x = 90, \beta_x = 90°, \gamma_x = 0° \rightarrow l_x = 0, m_x = 0, n_x = 1$$
$$\alpha_y = 0, \beta_y = 90, \gamma_y = 90 \rightarrow l_y = 1, m_y = 0; n_y = 0$$
$$\alpha_z = 90, \beta_z = 0, \gamma_z = 90 \rightarrow l_z = 0, m_z = 1, n_z = 0$$

$$[T1]^1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

$$The\ reduced\ matrix\ [T]\ is\ then:\ [T]^{*1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix};$$

$$[k]^1 = 10^2 \begin{bmatrix} 9600 & 0 & 0 & 0 & 0 & 0 & -9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 & 0 & 192 & 0 & -96 & 0 & 0 & 0 & 192 \\ 0 & 0 & 54 & 0 & -108 & 0 & 0 & 0 & -54 & 0 & -108 & 0 \\ 0 & 0 & 0 & 64.8 & 0 & 0 & 0 & 0 & 0 & -64.8 & 0 & 0 \\ 0 & 192 & 0 & 0 & 0 & 512 & 0 & -192 & 0 & 0 & 0 & 256 \\ 0 & 0 & -108 & 0 & 288 & 0 & 0 & 0 & 108 & 0 & 144 & 0 \\ -9600 & 0 & 0 & 0 & 0 & 0 & 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & -96 & 0 & 0 & 0 & -192 & 0 & 96 & 0 & 0 & 0 & -192 \\ 0 & 0 & -54 & 0 & 108 & 0 & 0 & 0 & 54 & 0 & 108 & 0 \\ 0 & 0 & 0 & -64.8 & 0 & 0 & 0 & 0 & 0 & 64.8 & 0 & 0 \\ 0 & 0 & -108 & 0 & 144 & 0 & 0 & 0 & 108 & 0 & 288 & 0 \\ 0 & 192 & 0 & 0 & 0 & 256 & 0 & -192 & 0 & 0 & 0 & 512 \end{bmatrix}$$

The reduced matrix $[\mathrm{k}]^1$ is then $[\mathrm{k}]^{*1}$

$$[k]^{*1} = 10^2 \begin{bmatrix} 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 & 0 & -192 \\ 0 & 0 & 54 & 0 & 108 & 0 \\ 0 & 0 & 0 & 64.8 & 0 & 0 \\ 0 & 0 & 108 & 0 & 288 & 0 \\ 0 & -192 & 0 & 0 & 0 & 512 \end{bmatrix}$$

$$[K]^{*1} = 10^2 \begin{bmatrix} 96 & 0 & 0 & 0 & -192 & 0 \\ 0 & 54 & 0 & 108 & 0 & 0 \\ 0 & 0 & 9600 & 0 & 0 & 0 \\ 0 & 108 & 0 & 288 & 0 & 0 \\ -192 & 0 & 0 & 0 & 512 & 0 \\ 0 & 0 & 0 & 0 & 0 & 64.8 \end{bmatrix}$$

Element 2: $L = 4, A = 0.12, I_{y'} = 0.0016, I_{z'} = 0.0009, \;\; J = 0.001994$

$$\alpha_x = 0, \beta_x = 90°, \gamma_x = 90° \;\rightarrow l_x = 1, m_x = 0, n_x = 0$$
$$\alpha_y = 90, \beta_y = 0, \gamma_y = 90 \rightarrow l_y = 0, m_y = 1; n_y = 0$$
$$\alpha_z = 90, \beta_z = 90, \gamma_z = 0 \rightarrow l_z = 0, m_z = 0, n_z = 1$$

$$[T]^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; [T]^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]^2 = 10^2 \begin{matrix} 0 & 54 & 0 & 0 & 0 & 108 & 0 & -54 & 0 & 0 & 0 & 108 \\ 0 & 0 & 96 & 0 & -192 & 0 & 0 & 0 & -96 & 0 & -192 & 0 \\ 0 & 0 & 0 & 65 & 0 & 0 & 0 & 0 & 0 & -65 & 0 & 0 \\ 0 & 0 & -192 & 0 & 512 & 0 & 0 & 0 & 192 & 0 & 256 & 0 \\ 0 & 108 & 0 & 0 & 0 & 288 & 0 & -108 & 0 & 0 & 0 & 144 \\ -9600 & 0 & 0 & 0 & 0 & 0 & 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & -54 & 0 & 0 & 0 & -108 & 0 & 54 & 0 & 0 & 0 & -108 \\ 0 & 0 & -96 & 0 & 192 & 0 & 0 & 0 & 96 & 0 & 192 & 0 \\ 0 & 0 & 0 & -65 & 0 & 0 & 0 & 0 & 0 & 65 & 0 & 0 \\ 0 & 0 & -192 & 0 & 256 & 0 & 0 & 0 & 192 & 0 & 512 & 0 \\ 0 & 108 & 0 & 0 & 0 & 144 & 0 & -108 & 0 & 0 & 0 & 288 \end{matrix}$$

$$[k]^{*2} = 10^2 \begin{bmatrix} 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 108 \\ 0 & 0 & 96 & 0 & -192 & 0 \\ 0 & 0 & 0 & 64.8 & 0 & 0 \\ 0 & 0 & -192 & 0 & 512 & 0 \\ 0 & 108 & 0 & 0 & 0 & 288 \end{bmatrix}$$

$$[K]^{*2} = 10^2 \begin{matrix} 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 108 \\ 0 & 0 & 96 & 0 & -192 & 0 \\ 0 & 0 & 0 & 64.8 & 0 & 0 \\ 0 & 0 & -192 & 0 & 512 & 0 \\ 0 & 108 & 0 & 0 & 0 & 288 \end{matrix}$$

element 3: $L = 4, A = 0.12, I_{y'} = 0.0016, I_{z'} = 0.0009, \quad J = 0.001994$

$$\alpha_x = 90, \beta_x = 0°, \gamma_x = 90° \rightarrow l_x = 0, m_x = 1, n_x = 0$$

$$\alpha_y = 180, \beta_y = 90, \gamma_y = 90 \rightarrow l_y = -1, m_y = 0; n_y = 0$$

$$\alpha_z = 90, \beta_z = 90, \gamma_z = 0 \rightarrow l_z = 0, m_z = 0, n_z = 1$$

$$[T1]^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [T]^{*3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]^3 = 10^2 \begin{bmatrix} 9600 & 0 & 0 & 0 & 0 & 0 & -9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 108 & 0 & -54 & 0 & 0 & 0 & 108 \\ 0 & 0 & 96 & 0 & -192 & 0 & 0 & 0 & -96 & 0 & -192 & 0 \\ 0 & 0 & 0 & 65 & 0 & 0 & 0 & 0 & 0 & -65 & 0 & 0 \\ 0 & 0 & -192 & 0 & 512 & 0 & 0 & 0 & 192 & 0 & 256 & 0 \\ 0 & 108 & 0 & 0 & 0 & 288 & 0 & -108 & 0 & 0 & 0 & 144 \\ -9600 & 0 & 0 & 0 & 0 & 0 & 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & -54 & 0 & 0 & 0 & -108 & 0 & 54 & 0 & 0 & 0 & -108 \\ 0 & 0 & -96 & 0 & 192 & 0 & 0 & 0 & 96 & 0 & 192 & 0 \\ 0 & 0 & 0 & -65 & 0 & 0 & 0 & 0 & 0 & 65 & 0 & 0 \\ 0 & 0 & -192 & 0 & 256 & 0 & 0 & 0 & 192 & 0 & 512 & 0 \\ 0 & 108 & 0 & 0 & 0 & 144 & 0 & -108 & 0 & 0 & 0 & 288 \end{bmatrix}$$

$$[k]^{*3} = 10^2 \begin{bmatrix} 9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 0 & 0 & 0 & 108 \\ 0 & 0 & 96 & 0 & -192 & 0 \\ 0 & 0 & 0 & 64.8 & 0 & 0 \\ 0 & 0 & -192 & 0 & 512 & 0 \\ 0 & 108 & 0 & 0 & 0 & 288 \end{bmatrix}$$

$$[K]^{*3} = 10^2 \begin{bmatrix} 54 & 0 & 0 & 0 & 0 & -108 \\ 0 & 9600 & 0 & 0 & 0 & 0 \\ 0 & 0 & 96 & 192 & 0 & 0 \\ 0 & 0 & 192 & 512 & 0 & 0 \\ 0 & 0 & 0 & 0 & 64.8 & 0 \\ -108 & 0 & 0 & 0 & 0 & 288 \end{bmatrix}$$

Assembly:

Since the reduced stiffness matrices of the three elements involve the same degrees of freedom, then:

$$[K]^* = [k]^{*1} + [k]^{*2} + [k]^{*3}$$

$$[K]^* = 10^2 \begin{bmatrix} 9750 & 0 & 0 & 0 & -192 & -108 \\ 0 & 9708 & 0 & 108 & 0 & 108 \\ 0 & 0 & 9792 & 192 & -192 & 0 \\ 0 & 108 & 192 & 864.8 & 0 & 0 \\ -192 & 0 & -192 & 0 & 1088.8 & 0 \\ -108 & 108 & 0 & 0 & 0 & 640.8 \end{bmatrix}$$

Formation of the Reduced Force Vector

Reduced Nodal Forces:

$$\{fn\}^* = \begin{Bmatrix} 20 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Equivalent Nodal Loads for Element 2 (Flexion in the x-z Plane):

$$\{feq\}^{f2} = \begin{Bmatrix} T_{y_1'} \\ M_{z_1'} \\ T_{y_2'} \\ M_{z_2'} \end{Bmatrix}^2 = \int_0^L \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ -\frac{x^2}{L} + \frac{x^3}{L^2} \end{bmatrix} \times 12.dx = \begin{bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{L^3} \\ \frac{x^2}{2} - \frac{2x^3}{3L} + \frac{x^4}{4L^2} \\ \frac{x^3}{L^2} - \frac{x^4}{2L^3} \\ -\frac{x^3}{3L} + \frac{x^4}{4L^2} \end{bmatrix}_0^L = \begin{Bmatrix} 24 \\ 16 \\ 24 \\ -16 \end{Bmatrix}$$

$$\{feq\}^{*2} = \begin{Bmatrix} 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 16 \end{Bmatrix} \rightarrow \{Feq\}^2 = [T]^{*2^T}\{\{feq\}^{*2}\} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 16 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 16 \end{Bmatrix}$$

Equivalent Nodal Loads for Element 3 (Flexion in the y-z Plane)

$$\{feq\}^{f3} = \begin{Bmatrix} T_{z_1'} \\ M_{y_1'} \\ T_{z_2'} \\ M_{y_2'} \end{Bmatrix}^3 = \int_0^L \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ -x + \frac{2x^2}{L} - \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{x^2}{L} - \frac{x^3}{L^2} \end{bmatrix} \times (-24).dx = \begin{bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{L^3} \\ -\frac{x^2}{2} + \frac{2x^3}{3L} - \frac{x^4}{4L^2} \\ \frac{x^3}{L^2} - \frac{x^4}{2L^3} \\ \frac{x^3}{3L} - \frac{x^4}{4L^2} \end{bmatrix}_0^L = \begin{Bmatrix} -48 \\ 32 \\ -48 \\ -32 \end{Bmatrix}$$

$$\{feq\}^{*3} = \begin{Bmatrix} 0 \\ 0 \\ -48 \\ 0 \\ 32 \\ 0 \end{Bmatrix} \rightarrow \{Feq\}^3 = [T]^{*3^T}\{\{feq\}^{*3}\} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -48 \\ 0 \\ 32 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -48 \\ -32 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{F\}^* = \{fn\}^{*+}\{Feq\}^2 + \{Feq\}^3 = \begin{Bmatrix} 20 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 16 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -48 \\ -32 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 34 \\ -48 \\ -32 \\ 0 \\ 16 \end{Bmatrix}$$

Determination of displacements at node 2;

$$\begin{Bmatrix} 20 \\ 34 \\ -48 \\ -32 \\ 0 \\ 16 \end{Bmatrix} = \begin{bmatrix} 9750 & 0 & 0 & 0 & -192 & -108 \\ 0 & 9708 & 0 & 108 & 0 & 108 \\ 0 & 0 & 9792 & 192 & -192 & 0 \\ 0 & 108 & 192 & 864.8 & 0 & 0 \\ -192 & 0 & -192 & 0 & 1088.8 & 0 \\ -108 & 108 & 0 & 0 & 0 & 640.8 \end{bmatrix} \begin{Bmatrix} D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \end{Bmatrix} = 10^{-3} \begin{Bmatrix} 0.0232 \\ 0.0363 \\ -0.0419 \\ -0.3653 \\ -0.0033 \\ 0.2475 \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{D_7} \\ \boldsymbol{D_8} \\ \boldsymbol{D_9} \\ \boldsymbol{D_{10}} \\ \boldsymbol{D_{11}} \\ \boldsymbol{D_{12}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{u_2} \\ \boldsymbol{v_2} \\ \boldsymbol{w_2} \\ \boldsymbol{\theta_{x2}} \\ \boldsymbol{\theta_{y2}} \\ \boldsymbol{\theta_{z2}} \end{Bmatrix} = \mathbf{10^{-3}} \begin{Bmatrix} \mathbf{0.0232} \\ \mathbf{0.0363} \\ \mathbf{-0.0419} \\ \mathbf{-0.3653} \\ \mathbf{-0.0033} \\ \mathbf{0.2475} \end{Bmatrix}$$

Forces at the Nodes of the Elements:

Forces at the Nodes of the Element 1: column:

$$\{D\}^1=\begin{Bmatrix}D_1\\D_2\\D_3\\D_4\\D_5\\D_6\\D_7\\D_8\\D_9\\D_{10}\\D_{11}\\D_{12}\end{Bmatrix}=10^{-3}\begin{Bmatrix}0\\0\\0\\0\\0\\0\\0.0232\\0.0363\\-0.0419\\-0.3653\\-0.0033\\0.2475\end{Bmatrix}\rightarrow\{d\}^1=[T]^1\{D\}^1=10^{-3}\begin{Bmatrix}0\\0\\0\\0\\0\\0\\-0.0419\\0.0232\\0.0363\\0.2475\\-0.3653\\-0.0033\end{Bmatrix}$$

$$\{fn\}^1=[k]^1\{d\}^1-\{feq\}^1\rightarrow\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\\F_6\\F_7\\F_8\\F_9\\F_{10}\\F_{11}\\F_{12}\end{Bmatrix}^1=\begin{Bmatrix}\boldsymbol{f_1}\\\boldsymbol{T_{y1}}\\\boldsymbol{T_{z1}}\\\boldsymbol{M_{x1}}\\\boldsymbol{M_{y1}}\\\boldsymbol{M_{z1}}\\\boldsymbol{f_2}\\\boldsymbol{T_{y2}}\\\boldsymbol{T_{z2}}\\\boldsymbol{M_{x2}}\\\boldsymbol{M_{y2}}\\\boldsymbol{M_{z2}}\end{Bmatrix}^1=\begin{Bmatrix}40.25\\-0.29\\3.75\\-1.60\\-4.87\\-0.53\\-40.25\\0.29\\-3.75\\1.60\\-10.13\\-0.61\end{Bmatrix}$$

Forces at the Nodes of the Element 2: Beam//x :

$$\{D\}^2=\begin{Bmatrix}D_7\\D_8\\D_9\\D_{10}\\D_{11}\\D_{12}\\D_{13}\\D_{14}\\D_{15}\\D_{16}\\D_{17}\\D_{18}\end{Bmatrix}=10^{-3}\begin{Bmatrix}0.0232\\0.0363\\-0.0419\\-0.3653\\-0.0033\\0.2475\\0\\0\\0\\0\\0\\0\end{Bmatrix}\rightarrow\{d\}^2=[T]^2\{D\}^2=10^{-3}\begin{Bmatrix}0.0232\\0.0363\\-0.0419\\-0.3653\\-0.0033\\0.2475\\0\\0\\0\\0\\0\\0\end{Bmatrix}$$

$$\{fn\}^2=[k]^2\{d\}^2-\{feq\}^2\rightarrow\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\\F_6\\F_7\\F_8\\F_9\\F_{10}\\F_{11}\\F_{12}\end{Bmatrix}^2=\begin{Bmatrix}\boldsymbol{f_1}\\\boldsymbol{T_{y1}}\\\boldsymbol{T_{z1}}\\\boldsymbol{M_{x1}}\\\boldsymbol{M_{y1}}\\\boldsymbol{M_{z1}}\\\boldsymbol{f_2}\\\boldsymbol{T_{y2}}\\\boldsymbol{T_{z2}}\\\boldsymbol{M_{x2}}\\\boldsymbol{M_{y2}}\\\boldsymbol{M_{z2}}\end{Bmatrix}^2=\begin{Bmatrix}22.26\\-21.13\\-0.34\\-2.37\\0.64\\-8.48\\-22.26\\-26.87\\0.34\\2.37\\0.72\\19.96\end{Bmatrix}$$

Forces at the Nodes of the Element 3: Beam//y:

$$\{D\}^3=\begin{Bmatrix}D_7\\D_8\\D_9\\D_{10}\\D_{11}\\D_{12}\\D_{19}\\D_{20}\\D_{21}\\D_{22}\\D_{23}\\D_{24}\end{Bmatrix}=10^{-3}\begin{Bmatrix}0.0232\\0.0363\\-0.0419\\-0.3653\\-0.0033\\0.2475\\0\\0\\0\\0\\0\\0\end{Bmatrix}\rightarrow\{d\}^3=[T]^3\{D\}^3=10^{-3}\begin{Bmatrix}0.0232\\0.0363\\-0.0419\\-0.3653\\-0.0033\\0.2475\\0\\0\\0\\0\\0\\0\end{Bmatrix}$$

$$\{fn\}^3=[k]^3\{d\}^3-\{feq\}^3\rightarrow\begin{Bmatrix}F_1\\F_2\\F_3\\F_4\\F_5\\F_6\\F_7\\F_8\\F_9\\F_{10}\\F_{11}\\F_{12}\end{Bmatrix}^3=\begin{Bmatrix}\boldsymbol{f_1}\\\boldsymbol{T_{y1}}\\\boldsymbol{T_{z1}}\\\boldsymbol{M_{x1}}\\\boldsymbol{M_{y1}}\\\boldsymbol{M_{z1}}\\\boldsymbol{f_2}\\\boldsymbol{T_{y2}}\\\boldsymbol{T_{z2}}\\\boldsymbol{M_{x2}}\\\boldsymbol{M_{y2}}\\\boldsymbol{M_{z2}}\end{Bmatrix}^3=\begin{Bmatrix}34.88\\2.55\\40.58\\-0.02\\-12.49\\6.88\\-34.88\\-2.55\\55.42\\0.02\\42.16\\3.31\end{Bmatrix}$$