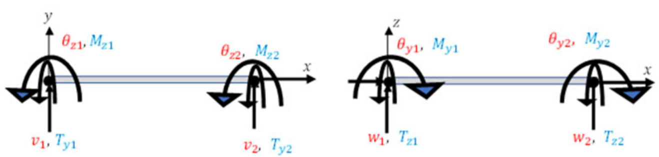
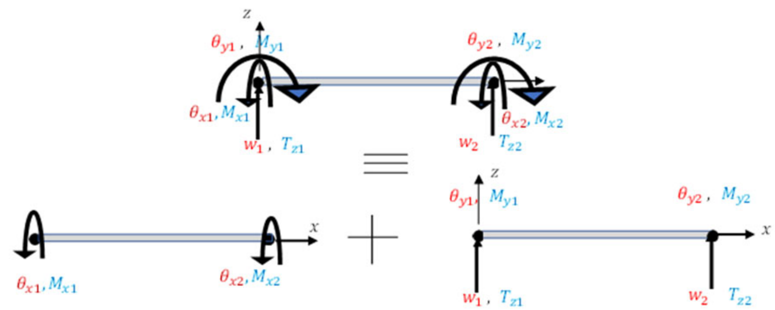
2.5. : Bending and Torsion Bar Element (crossed beam element) and crossed beam structures

2.5.1 : Definition, Notations and sign conventions



Case of bending in the x-z plane: Common case of crossed beams



Element Stiffness Matrix

Element Basic equations

The displacement field of this element is the rotation around the local axis $\boldsymbol{x}$: $(\boldsymbol{\theta_x})$

The deformation field is reduced to the angular deformation $\boldsymbol{\gamma_{yz} = \gamma_{yz}(x,\rho) = \rho \frac{d\theta_x}{dx}}$

The stress field is reduced to the tangential stress $\boldsymbol{\tau_{yz} = \tau_{yz}(x,\rho) = G\gamma(x,\rho) = \frac{d\theta_x}{dx}}$

$$\boldsymbol{M_x = \int_A \rho\tau_{yz}(x,\rho)\,dA = \int_A \rho G\rho \frac{d\theta_x}{dx}\,dA = G\frac{d\theta_x}{dx}\int_A \rho^2\,dA = GJ\frac{d\theta_x}{dx}}$$

By approaching $\boldsymbol{\theta_x}$ *by polynomials or by Lagrange interpolation we find:*

$$\boldsymbol{\theta_x = \left\lfloor 1-\frac{x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}}$$

$$\boldsymbol{\frac{d\theta_x}{dx} = \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}}$$

$$\boldsymbol{M_{x1} = -GJ\left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}}$$

$$\boldsymbol{M_{x2} = GJ\left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}}$$

$$\boldsymbol{\begin{Bmatrix}M_{x1}\\ M_{x2}\end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix} = \frac{EJ}{2L(1+\upsilon)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}}$$

So, for the torsion bar element:

$$[k] = \frac{EJ}{2L(1+v)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The bending and torsion bar element is a superposition of the torsion bar element and the bending element.

$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{EJ}{2L(1+v)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} \quad + \quad \begin{Bmatrix} T_{z1} \\ M_{y1} \\ T_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_y}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$

$$\{f\} = \lfloor T_{z1} \quad M_{x1} \quad M_{y1} \quad T_{z2} \quad M_{x2} \quad M_{y2} \rfloor^T \; ; \; \{d\} = \lfloor w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \rfloor^T$$

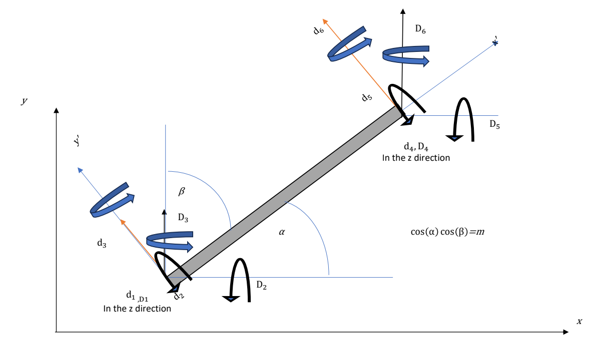
$$\{f\} = [k]\{d\} \quad \rightarrow$$

$$\begin{Bmatrix} T_{z1} \\ M_{x1} \\ M_{y1} \\ T_{z2} \\ M_{x2} \\ M_{y2} \end{Bmatrix} = \frac{E}{L}\begin{bmatrix} \frac{12I_y}{L^2} & 0 & \frac{-6I_y}{L} & \frac{-12I_y}{L^2} & 0 & \frac{-6I_y}{L} \\ 0 & \frac{J}{2(1+v)} & 0 & 0 & \frac{-J}{2(1+v)} & 0 \\ \frac{-6I_y}{L} & 0 & 4I_y & \frac{6I_y}{L} & 0 & 2I_y \\ \frac{-12I}{L^2} & 0 & \frac{6I}{L} & \frac{12I_y}{L^2} & 0 & \frac{6I_y}{L} \\ 0 & \frac{-J}{2(1+v)} & 0 & 0 & \frac{J}{2(1+v)} & 0 \\ \frac{-6I}{L} & 0 & 2I_y & \frac{6I_y}{L} & 0 & 4I_y \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \end{Bmatrix}$$

So, for the bending and torsion bar element and considering that the bar is in the x-y plane (bending in the x-z plane):

$$[k] = \frac{E}{L}\begin{bmatrix} \frac{12I_y}{L^2} & 0 & \frac{-6I_y}{L} & \frac{-12I_y}{L^2} & 0 & \frac{-6I_y}{L} \\ 0 & \frac{J}{2(1+v)} & 0 & 0 & \frac{-J}{2(1+v)} & 0 \\ \frac{-6I_y}{L} & 0 & 4I_y & \frac{6I_y}{L} & 0 & 2I_y \\ \frac{-12I}{L^2} & 0 & \frac{6I}{L} & \frac{12I_y}{L^2} & 0 & \frac{6I_y}{L} \\ 0 & \frac{-J}{2(1+v)} & 0 & 0 & \frac{J}{2(1+v)} & 0 \\ \frac{-6I}{L} & 0 & 2I_y & \frac{6I_y}{L} & 0 & 4I_y \end{bmatrix}$$

The transformation matrix for this element is obtained as follows:



$$\text{projection on } z':\ d_1 = D_1 \text{ car } z' \parallel z$$

$$\text{projection on } x' \text{ at node 1:}\quad d_2 = D_2\cos(\alpha) + D_3\cos(\beta) = lD_1 + \mathrm{m}D_3$$

$$\text{projection on } x' \text{ at node 1:}\quad d_3 = -D_2\cos(\beta) + D_3\cos(\alpha) = -mD_1 + \mathrm{l}D_3$$

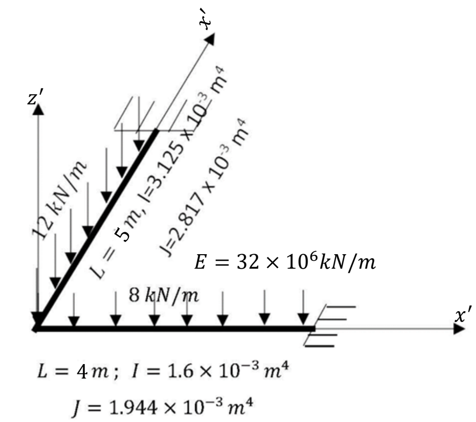
Likewise at node 2 for d_4 ; d_5 et d_6

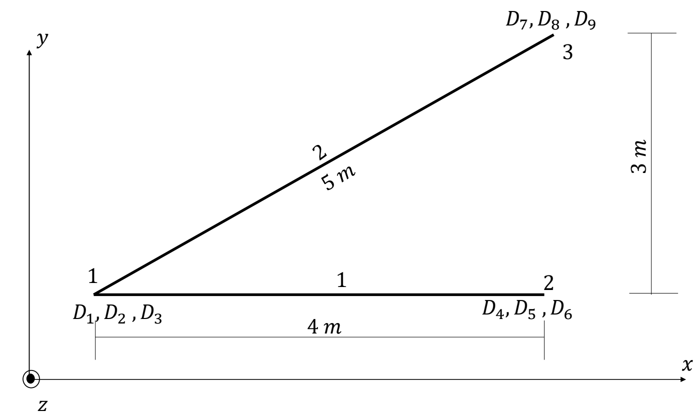
$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l & m & 0 & 0 & 0 \\ 0 & -m & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & l & m \\ 0 & 0 & 0 & 0 & -m & l \end{bmatrix}$$

The stiffness matrix of this element in global coordinate is:

$$[K] = \frac{E}{L}\begin{bmatrix} \frac{12I}{L^2} & \frac{6mI}{L} & \frac{-6lI}{L} & \frac{-12I}{L^2} & \frac{6mI}{L} & \frac{-6lI}{L} \\ \frac{6mI}{L} & \left(\frac{l^2J}{2(1+v)} + 4m^2I\right) & lm\left(\frac{J}{2(1+v)} - 4I\right) & \frac{-6mI}{L} & \left(\frac{-l^2J}{2(1+v)} + 2m^2I\right) & -lm\left(\frac{J}{2(1+v)} + 2I\right) \\ \frac{-6lI}{L} & lm\left(\frac{J}{2(1+v)} - 4I\right) & \left(\frac{m^2J}{2(1+v)} + 4l^2I\right) & \frac{6lI}{L} & -lm\left(\frac{J}{2(1+v)} + 2I\right) & \left(\frac{-m^2J}{2(1+v)} + 2l^2I\right) \\ \frac{-12I}{L^2} & \frac{-6mI}{L} & \frac{6lI}{L} & \frac{12I}{L^2} & \frac{-6mI}{L} & \frac{6lI}{L} \\ \frac{6mI}{L} & \left(\frac{-l^2J}{2(1+v)} + 2m^2I\right) & -lm\left(\frac{J}{2(1+v)} + 2I\right) & -\frac{6mI}{L} & \left(\frac{l^2J}{2(1+v)} + 4m^2I\right) & lm\left(\frac{J}{2(1+v)} - 4I\right) \\ \frac{-6lI}{L} & -lm\left(\frac{J}{2(1+v)} + 2I\right) & \left(\frac{-m^2J}{2(1+v)} + 2l^2I\right) & \frac{6lI}{L} & lm\left(\frac{J}{2(1+v)} - 4I\right) & \left(\frac{m^2J}{2(1+v)} + 4l^2I\right) \end{bmatrix}$$

Example:





Continuity of displacements:

$$\{D\} = \lfloor D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \quad D_7 \quad D_8 \quad D_9 \rfloor^T$$

$$\{D\} = \lfloor w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \rfloor^T$$

$$\{D\}^1 = \lfloor D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6 \rfloor^T$$

$$\{D\}^2 = \lfloor D_1 \quad D_2 \quad D_3 \quad D_7 \quad D_8 \quad D_9 \rfloor^T$$

Boundary conditions:

$$D_1 = \quad D_2 = \quad D_3 = D_7 = \quad D_8 = \quad D_9 = 0$$

Equivalent nodal loads:

Element 1: nodal loads equivalent to the transverse load (bending load)

$$\{f_{eq}\}_f^1 = \int_0^4 \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ -x + \frac{2x^2}{L} - \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{x^2}{L} - \frac{x^3}{L^2} \end{bmatrix} \times (-8).dx = -8 \times \begin{bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{2L^3} \\ -\frac{x^2}{2} + \frac{2x^3}{3L} - \frac{x^4}{4L^2} \\ \frac{3x^3}{2L^2} - \frac{x^4}{2L^3} \\ -\frac{x^3}{3L} - \frac{x^4}{4L^2} \end{bmatrix}_0^4 = \begin{Bmatrix} -16 \\ 10.67 \\ -16 \\ -10.67 \end{Bmatrix}$$

$$\{f_{eq}\}_T^1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{f_{eq}\}^1 = \begin{Bmatrix} -16 \\ 0 \\ 10.67 \\ -16 \\ 0 \\ -10.67 \end{Bmatrix}$$

$$\{F_{eq}\}^1 = [T]^{1^T}\{f_{eq}\}^1 = \begin{Bmatrix} -16 \\ 0 \\ 10.67 \\ -16 \\ 0 \\ -10.67 \end{Bmatrix}$$

Element 2: nodal loads equivalent to the transverse load (bending load)

$$\{f_{eq}\}_f^2 = \int_0^5 \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ -x + \frac{2x^2}{L} - \frac{x^3}{L^2} \\ \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ \frac{x^2}{L} - \frac{x^3}{L^2} \end{bmatrix} \times (-12).dx = -12 \times \begin{bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{2L^3} \\ -\frac{x^2}{2} + \frac{2x^3}{3L} - \frac{x^4}{4L^2} \\ \frac{3x^3}{2L^2} - \frac{x^4}{2L^3} \\ -\frac{x^3}{3L} - \frac{x^4}{4L^2} \end{bmatrix}_0^5 = \begin{Bmatrix} -30 \\ 25 \\ -30 \\ -25 \end{Bmatrix}$$

$$\{f_{eq}\}_T^2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{f_{eq}\}^2 = \begin{Bmatrix} -30 \\ 0 \\ 25 \\ -30 \\ 0 \\ -25 \end{Bmatrix}$$

$$\{F_{eq}\}^2 = [T]^{2^T}\{f_{eq}\}^2 = \begin{Bmatrix} -30 \\ -15 \\ 20 \\ -30 \\ 15 \\ -20 \end{Bmatrix}$$

Element 1 :

$$\boldsymbol{E = 32 \times 10^6 kN/m;\ v = 0.2; I = 0.0016\ m^4; J = 1.944\ m^4;\ L = 4m, l = 1\ ;\ m = 0}$$

$$[\boldsymbol{T}]^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\boldsymbol{k}]^1 = [\boldsymbol{K}]^1 = \begin{bmatrix} 9600 & 0 & -19200 & -9600 & 0 & -19200 \\ 0 & 6480 & 0 & 0 & -6480 & 0 \\ -19200 & 0 & 51200 & 19200 & 0 & 25600 \\ -9600 & 0 & 19200 & 9600 & 0 & 19200 \\ 0 & -6480 & 0 & 0 & 6480 & 0 \\ -19200 & 0 & 25600 & 19200 & 0 & 51200 \end{bmatrix}$$

Element 2 :

$E = 32 \times 10^6 kN/m;\ v = 0.2;\ I = 0.0016\ m^4;\ J = 2.817\ m^4;\ L = 5m, l = 0.8\ ;\ m = 0.6$

$$[T]^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & -0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.6 \\ 0 & 0 & 0 & 0 & -0.6 & 0.8 \end{bmatrix}$$

$$[k]^2 = \begin{bmatrix} 9600 & 0 & -24000 & -9600 & 0 & -24000 \\ 0 & 7513 & 0 & 0 & -7513 & 0 \\ -24000 & 0 & 80000 & 24000 & 0 & 40000 \\ -9600 & 0 & 24000 & 9600 & 0 & 24000 \\ 0 & -7513 & 0 & 0 & 7513 & 0 \\ -24000 & 0 & 40000 & 24000 & 0 & 80000 \end{bmatrix}$$

$$[K]^2 = \begin{bmatrix} 9600 & 14400 & -19200 & -9600 & 14400 & -19200 \\ 14400 & 33608 & -34794 & -14400 & 9592 & -22806 \\ -19200 & -34794 & 53905 & 19200 & -22806 & 22895 \\ -9600 & -14400 & 19200 & 9600 & -14400 & 19200 \\ 14400 & 9592 & -22806 & -14400 & 33608 & -34794 \\ -19200 & -22806 & 22895 & 19200 & -34794 & 53905 \end{bmatrix}$$

To simplify the calculations and lighten the writing of the assembly, let's introduce the boundary conditions into the elementary systems:

$$\{D\}^{*1} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \quad \{D\}^{*2} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \quad \{D\}^{*} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$

$$\{F_{eq}\}^{*1} = \begin{Bmatrix} -16 \\ 0 \\ 10.67 \end{Bmatrix} \quad \{F_{eq}\}^{*2} = \begin{Bmatrix} -30 \\ -15 \\ 20 \end{Bmatrix} \quad \{F_{eq}\}^{*} = \begin{Bmatrix} -46 \\ -15 \\ 30.67 \end{Bmatrix}$$

	9600 **9600**	**0** **14400**	**−19200** **−19200**
$[K]^* =$	**0** **14400**	**6480** **33608**	**0** **−34794**
	−19200 **−19200**	**0** **−34794**	**51200** **53905**

$$\begin{Bmatrix} -46 \\ -15 \\ 30.67 \end{Bmatrix} = \begin{bmatrix} 19200 & 14400 & -38400 \\ 14400 & 40088 & -34794 \\ -38400 & -34794 & 105105 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_{x1} \\ D_{y1} \end{Bmatrix} = \begin{Bmatrix} -0.006805 \\ 0.0002324 \\ -0.002117 \end{Bmatrix}$$

Forces in the Elements:

$$\{d\}^1 = [T]^1\{D\}^1 = \{D\}^1 = \begin{Bmatrix} -0.006805 \\ 0.0002324 \\ -0.002117 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f_n\}^1 = [k]^1\{d\}^1 - \{f_{eq}\}^1$$

$$\begin{Bmatrix} T_{z1} \\ M_{x1} \\ M_{y1} \\ T_{z2} \\ M_{x2} \\ M_{y2} \end{Bmatrix}^1 = \begin{bmatrix} 9600 & 0 & -19200 & -9600 & 0 & -19200 \\ 0 & 6480 & 0 & 0 & -6480 & 0 \\ -19200 & 0 & 51200 & 19200 & 0 & 25600 \\ -9600 & 0 & 19200 & 9600 & 0 & 19200 \\ 0 & -6480 & 0 & 0 & 6480 & 0 \\ -19200 & 0 & 25600 & 19200 & 0 & 51200 \end{bmatrix} \begin{Bmatrix} -0.006805 \\ 0.0002324 \\ -0.002117 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -16 \\ 0 \\ 10.67 \\ -16 \\ 0 \\ 10.67 \end{Bmatrix}$$

$$\{f_n\}^2 = [k]^2\{d\}^2 - \{f_{eq}\}^2$$

$$\{D\}^2 = \begin{Bmatrix} -0.006805 \\ 0.0002324 \\ -0.002117 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \ \{d\}^2 = [T]^2\{D\}^2 = \begin{Bmatrix} -0.006805 \\ 0.001085 \\ -0.001834 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} T_{z1} \\ M_{x1} \\ M_{y1} \\ T_{z2} \\ M_{x2} \\ M_{y2} \end{Bmatrix}^2 = \begin{bmatrix} 9600 & 0 & -24000 & -9600 & 0 & -24000 \\ 0 & 7513 & 0 & 0 & -7513 & 0 \\ -24000 & 0 & 80000 & 24000 & 0 & 40000 \\ -9600 & 0 & 24000 & 9600 & 0 & 24000 \\ 0 & -7513 & 0 & 0 & 7513 & 0 \\ -24000 & 0 & 40000 & 24000 & 0 & 80000 \end{bmatrix} \begin{Bmatrix} -0.006805 \\ 0.001085 \\ -0.001834 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -30 \\ 0 \\ 25 \\ -16 \\ 0 \\ -25 \end{Bmatrix}$$

$$\begin{Bmatrix} T_{z1} \\ M_{x1} \\ M_{y1} \\ T_{z2} \\ M_{x2} \\ M_{y2} \end{Bmatrix}^2 = \begin{Bmatrix} -8.67 \\ -8.15 \\ -8.36 \\ 51.33 \\ 8.15 \\ 114.99 \end{Bmatrix}$$