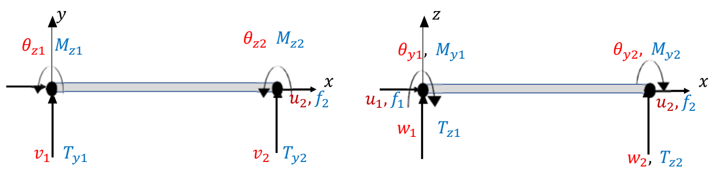
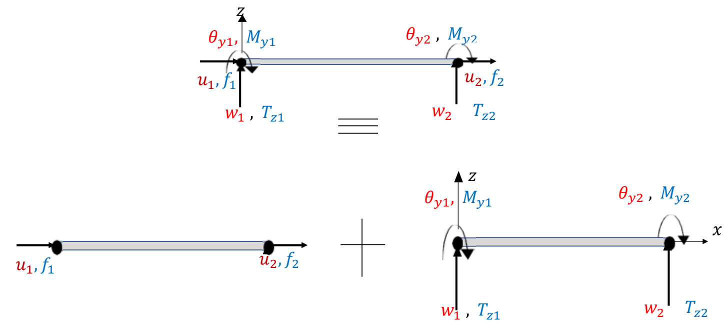
## *3.4 Bending and tensile- compression bar element (2D Frame element))*




### *Case of the bending in the x _ z plane*

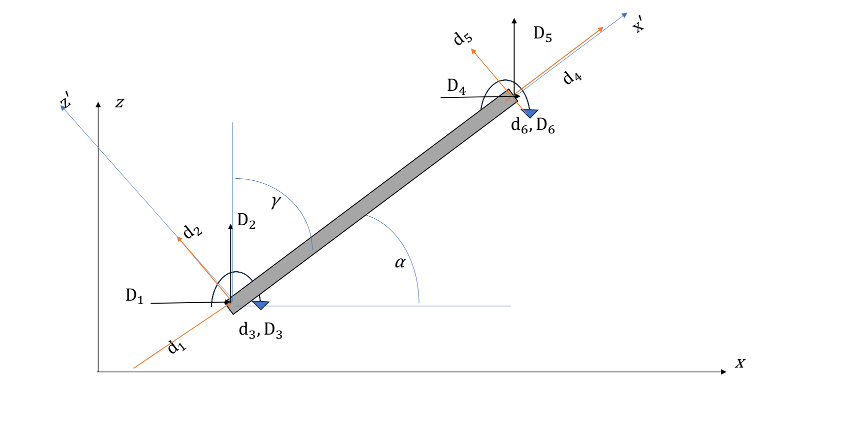



### *3-4-1. Standard 2D frame element*

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} T_{z1} \\ M_{y1} \\ T_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_y}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$

$$\{f\} = [k]\{d\}$$

$$\begin{Bmatrix} f_1 \\ T_{z1} \\ M_{y1} \\ f_2 \\ T_{z2} \\ M_{y2} \end{Bmatrix} = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I_y}{L^2} & \frac{-6I_y}{L} & 0 & -\frac{12I_y}{L^2} & \frac{-6I_y}{L} \\ 0 & \frac{-6I_y}{L} & 4I & 0 & \frac{6I_y}{L} & 2I \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & -\frac{12I_y}{L^2} & \frac{6I_y}{L} & 0 & \frac{12I_y}{L^2} & \frac{6I_y}{L} \\ 0 & \frac{-6I_y}{L} & 2I & 0 & \frac{6I_y}{L} & 4I \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ \theta_{y1} \\ u_2 \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$




$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ w_1 \\ \theta_{y1} \\ u_2 \\ w_2 \\ \theta_{y2} \end{Bmatrix}$$

$$[T] = \begin{bmatrix} l & n & 0 & 0 & 0 & 0 \\ -n & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & n & 0 \\ 0 & 0 & 0 & -n & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$I = I_y , \theta = \theta_y$

$$[K] = \frac{E}{L}\begin{bmatrix}
+\left(Al^2+\frac{12I}{L^2}n^2\right) & +ln\left(A-\frac{12I}{L^2}\right) & +\frac{6nI}{L} & -\left(Al^2+\frac{12I}{L^2}n^2\right) & -ln\left(A-\frac{12I}{L^2}\right) & +\frac{6nI}{L} \\
+ln\left(A-\frac{12I}{L^2}\right) & +\left(An^2+\frac{12I}{L^2}l^2\right) & -\frac{6lI}{L} & -ln\left(A-\frac{12I}{L^2}\right) & -\left(An^2+\frac{12I}{L^2}l^2\right) & -\frac{6lI}{L} \\
+\frac{6nI}{L} & -\frac{6lI}{L} & 4I & -\frac{6nI}{L} & \frac{6lI}{L} & 2I \\
-\left(Al^2+\frac{12I}{L^2}n^2\right) & -ln\left(A-\frac{12I}{L^2}\right) & -\frac{6nI}{L} & +\left(Al^2+\frac{12I}{L^2}n^2\right) & +ln\left(A-\frac{12I}{L^2}\right) & -\frac{6nI}{L} \\
-ln\left(A-\frac{12I}{L^2}\right) & -\left(An^2+\frac{12I}{L^2}l^2\right) & \frac{6lI}{L} & +ln\left(A-\frac{12I}{L^2}\right) & +\left(An^2+\frac{12I}{L^2}l^2\right) & +\frac{6lI}{L} \\
+\frac{6nI}{L} & -\frac{6lI}{L} & 2I & -\frac{6nI}{L} & +\frac{6lI}{L} & 4I
\end{bmatrix}$$

***Case of the bending in the x _ y plane***

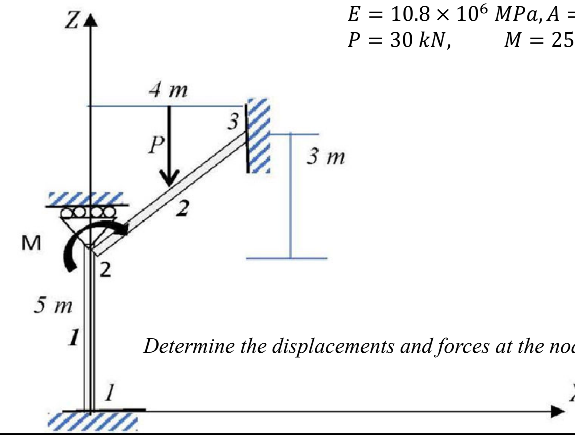
$$[k] = \frac{E}{L}\begin{bmatrix}
A & 0 & 0 & -A & 0 & 0 \\
0 & \frac{12I_z}{L^2} & \frac{6I_z}{L} & 0 & \frac{-12I_z}{L^2} & \frac{6I_z}{L} \\
0 & \frac{6I_z}{L} & 4I_z & 0 & \frac{-6I_z}{L} & 2I_z \\
-A & 0 & 0 & A & 0 & 0 \\
0 & \frac{-12I_z}{L^2} & \frac{-6I_z}{L} & 0 & \frac{12I_z}{L^2} & \frac{-6I_z}{L} \\
0 & \frac{6I_z}{L} & 2I_z & 0 & \frac{-6I_z}{L} & 4I_z
\end{bmatrix} , \quad [T] = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I = I_z \,, \theta = \theta_z$$

$$[K] = \frac{E}{L}\begin{bmatrix} +\left(Al^2 + \frac{12I}{L^2}m^2\right) & +lm\left(A - \frac{12I}{L^2}\right) & -\frac{6mI}{L} & -\left(Al^2 + \frac{12I}{L^2}m^2\right) & -lm\left(A - \frac{12I}{L^2}\right) & -\frac{6mI}{L} \\ +lm\left(A - \frac{12I}{L^2}\right) & +\left(Am^2 + \frac{12I}{L^2}l^2\right) & \frac{6lI}{L} & -lm\left(A - \frac{12I}{L^2}\right) & -\left(Am^2 + \frac{12I}{L^2}l^2\right) & \frac{6lI}{L} \\ -\frac{6mI}{L} & \frac{6lI}{L} & 4I & \frac{6mI}{L} & -\frac{6lI}{L} & 2I \\ -\left(Al^2 + \frac{12I}{L^2}m^2\right) & -lm\left(A - \frac{12I}{L^2}\right) & \frac{6mI}{L} & +\left(Al^2 + \frac{12I}{L^2}m^2\right) & +lm\left(A - \frac{12I}{L^2}\right) & \frac{6mI}{L} \\ -lm\left(A - \frac{12I}{L^2}\right) & -\left(Am^2 + \frac{12I}{L^2}l^2\right) & -\frac{6lI}{L} & +lm\left(A - \frac{12I}{L^2}\right) & +\left(Am^2 + \frac{12I}{L^2}l^2\right) & -\frac{6lI}{L} \\ -\frac{6mI}{L} & \frac{6lI}{L} & 2I & \frac{6mI}{L} & -\frac{6lI}{L} & 4I \end{bmatrix}$$

*Example:*

$E = 10.8 \times 10^6\ MPa, A = 0.12\ m^2$, I=0.0016 $m^4$
$P = 30\ kN, \qquad M = 25\ kN.m$




*Determine the displacements and forces at the nodes of the column and beam*

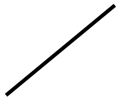
*Element 1 L=5m, l=0, n=1*

$$[T]^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[k]^1 = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I}{L^2} & \frac{-6I}{L} & 0 & \frac{-12I}{L^2} & \frac{-6I}{L} \\ 0 & \frac{-6I}{L} & 4I & 0 & \frac{6I}{L} & 2I \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{-12I}{L^2} & \frac{6I}{L} & 0 & \frac{12I}{L^2} & \frac{6I}{L} \\ 0 & \frac{-6I}{L} & 2I & 0 & \frac{6I}{L} & 4I \end{bmatrix}$$

$$K^1 = \frac{E}{L}\begin{bmatrix} \frac{12I}{L^2} & 0 & \frac{6I}{L} & -\frac{12I}{L^2} & 0 & \frac{6I}{L} \\ 0 & A & 0 & 0 & -A & 0 \\ \frac{6I}{L} & 0 & 4I & -\frac{6I}{L} & 0 & 2I \\ -\frac{12I}{L^2} & 0 & \frac{6I}{L} & \frac{12I}{L^2} & 0 & -\frac{6I}{L} \\ 0 & -A & 0 & 0 & A & 0 \\ \frac{6I}{L} & 0 & 2I & -\frac{6I}{L} & 0 & 4I \end{bmatrix}$$

*Element 2 L=5m, l=0.8, m=0.6*



$$[T]^2 = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]^2 = [k]^1 = \frac{E}{L}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{-6I}}{\boldsymbol{L}} & \boldsymbol{0} & \frac{\boldsymbol{-12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{-6I}}{\boldsymbol{L}} \\ \boldsymbol{0} & \frac{\boldsymbol{-6I}}{\boldsymbol{L}} & \boldsymbol{4I} & \boldsymbol{0} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{2I} \\ \boldsymbol{-A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\boldsymbol{-12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{0} & \frac{\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} \\ \boldsymbol{0} & \frac{\boldsymbol{-6I}}{\boldsymbol{L}} & \boldsymbol{2I} & \boldsymbol{0} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{4I} \end{bmatrix}$$

$$[\boldsymbol{K}]^2 = \frac{\boldsymbol{E}}{\boldsymbol{L}}\begin{bmatrix} \left(0.64\boldsymbol{A} + 0.36\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & -\left(0.64\boldsymbol{A} + 0.36\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} \\ 0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & \left(0.36\boldsymbol{A} + .64\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} & -0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -\left(0.36\boldsymbol{A} + 0.64\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} \\ 0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & -0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 4\boldsymbol{I} & -0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 2\boldsymbol{I} \\ -\left(0.64\boldsymbol{A} + 0.36\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & \left(0.64\boldsymbol{A} + 0.36\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} \\ -0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & -\left(0.36\boldsymbol{A} + 0.64\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 0.48\left(\boldsymbol{A} - \frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & \left(0.36\boldsymbol{A} + .64\frac{12\boldsymbol{I}}{\boldsymbol{L}^2}\right) & 0.8\frac{6\boldsymbol{lI}}{\boldsymbol{L}} \\ 0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & -0.8\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 2\boldsymbol{I} & -0.6\frac{6\boldsymbol{I}}{\boldsymbol{L}} & 0.8\frac{6\boldsymbol{lI}}{\boldsymbol{L}} & 4\boldsymbol{I} \end{bmatrix}$$

*Assembly:*

$$\{D\}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}; \quad \{D\}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}^2 = \begin{Bmatrix} D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix}; \quad \{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix}$$

| | $\frac{12I}{L^2}$ | 0 | $\frac{6I}{L}$ | $\frac{-12I}{L^2}$ | 0 | $\frac{6I}{L}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | $A$ | 0 | 0 | $-A$ | 0 | | | |
| | $\frac{6I}{L}$ | 0 | $4I$ | $\frac{-6I}{L}$ | 0 | $2I$ | | | |
| | $\frac{-12I}{L^2}$ | 0 | $\frac{6I}{L}$ | 0<br>$0.64A + 0.36\frac{12I}{L^2}$ | $\frac{12I}{L^2}$<br>$0.48\left(A - \frac{12I}{L^2}\right)$ | $\frac{6I}{L}$<br>$0.6\frac{6I}{L}$ | $-(0.64A + 0.36\frac{12I}{L^2})$ | $-0.48\left(A - \frac{12I}{L^2}\right)$ | $0.6\frac{6I}{L}$ |
| $K = \frac{E}{L}$ | 0 | $-A$ | 0 | $A$<br>$0.48\left(A - \frac{12I}{L^2}\right)$ | 0<br>$0.36A+.64\frac{12I}{L^2}$ | 0<br>$-0.8\frac{6I}{L}$ | $-0.48\left(A - \frac{12I}{L^2}\right)$ | $-(0.36A+.64\frac{12I}{L^2})$ | $-0.8\frac{6I}{L}$ |
| | 0 | $\frac{-6I}{L}$ | $2I$ | 0<br>$0.6\frac{6I}{L}$ | $\frac{6I}{L}$<br>$-0.8\frac{6I}{L}$ | $4I$<br>$4I$ | $-0.6\frac{6I}{L}$ | $0.8\frac{6I}{L}$ | $2I$ |
| | | | | $-(0.64A + 0.36\frac{12I}{L^2})$ | $-0.48\left(A - \frac{12I}{L^2}\right)$ | $-0.6\frac{6I}{L}$ | $0.64A + 0.36\frac{12I}{L^2}$ | $0.48\left(A - \frac{12I}{L^2}\right)$ | $-0.6\frac{6I}{L}$ |
| | | | | $-0.48\left(A - \frac{12I}{L^2}\right)$ | $-(0.36A+.64\frac{12I}{L^2})$ | $0.8\frac{6I}{L}$ | $0.48\left(A - \frac{12I}{L^2}\right)$ | $0.36A+.64\frac{12I}{L^2}$ | $0.8\frac{6I}{L}$ |
| | | | | $0.6\frac{6I}{L}$ | $-0.8\frac{6I}{L}$ | $2I$ | $-0.6\frac{6I}{L}$ | $0.8\frac{6I}{L}$ | $4I$ |

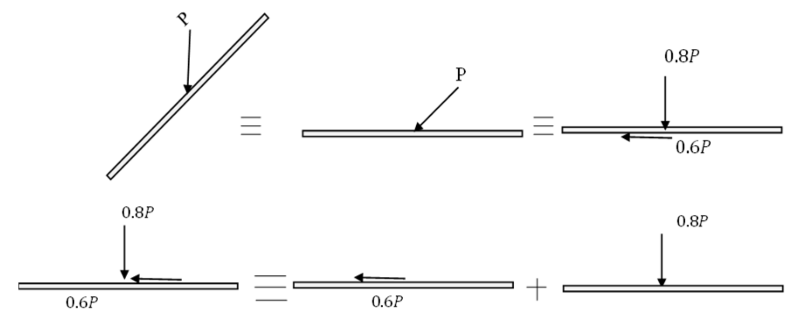
***Boundary conditions:***

$$\boldsymbol{D_1 = D_2 = D_3 = 0;\ D_5 = 0;\ D_7 = D_8 = D_9 = 0}$$

$$\boldsymbol{K = \frac{E}{L}\begin{bmatrix} 0.64A + 1.36\frac{12I}{L^2} & -0.4\frac{6I}{L} \\ -0.4\frac{6I}{L} & 8I \end{bmatrix}}$$

$$\boldsymbol{[K] = 10^5\begin{bmatrix} 1.6814 & -0.0166 \\ -0.0166 & 0.2765 \end{bmatrix}}$$

***Equivalent Load in Local Coordinates to the Concentrated Load on Element 2:***




$$\{f_{eq}\} = \{f_{eq}^1\} + \{f_{eq}^2\}$$

$$\{f_{eq}^1\} = \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix}_{x=\frac{L}{2}} \times (-0.6P) = \begin{Bmatrix} -0.3P \\ -0.3P \end{Bmatrix} = \begin{Bmatrix} -9 \\ -9 \end{Bmatrix}$$

$$\{f_{eq}^2\} = \begin{bmatrix} \boldsymbol{1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}} \\ \boldsymbol{-x + \frac{2x^2}{L} - \frac{x^3}{L^2}} \\ \boldsymbol{\frac{3x^2}{L^2} - \frac{2x^3}{L^3}} \\ \boldsymbol{\frac{x^2}{L} - \frac{x^3}{L^2}} \end{bmatrix}_{x=\frac{L}{2}} \times (-\,0.8P = \begin{Bmatrix} -0.4P \\ 0.5P \\ -0.4P \\ -0.5P \end{Bmatrix} = \begin{Bmatrix} -12 \\ 15 \\ -12 \\ -15 \end{Bmatrix}$$

$$\{f_{eq}\} = \begin{Bmatrix} -9 \\ -12 \\ 15 \\ -9 \\ -12 \\ -15 \end{Bmatrix}$$

$\{F_{eq}\}^2 = [T]^T\{f_{eq}\}^2$, *which gives:*

$$\{F_{eq}\}^2 = \begin{bmatrix} 0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -9 \\ -12 \\ 15 \\ -9 \\ -12 \\ -15 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -15 \\ 15 \\ 0 \\ -15 \\ -15 \end{Bmatrix}$$

$$\{F_{eq}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 + 0 \\ 0 + (-15) \\ 0 + 15 \\ 0 \\ -15 \\ -15 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -15 \\ 15 \\ 0 \\ -15 \\ -15 \end{Bmatrix}$$

*For the entire structure*

$$\{F\} = \begin{Bmatrix} R_{x1} \\ R_{z1} \\ M_{y1} \\ 0 \\ R_{z2} \\ 25 \\ R_{x3} \\ R_{z31} \\ M_{y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -15 \\ 15 \\ 0 \\ -15 \\ -15 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{R_{x1}} \\ \boldsymbol{R_{z1}} \\ \boldsymbol{M_{y1}} \\ 0 \\ \boldsymbol{R_{z2}} - 15 \\ 40 \\ \boldsymbol{R_{x3}} \\ \boldsymbol{R_{z31-15}} \\ \boldsymbol{M_{y3}} - 15 \end{Bmatrix}$$

*The Reduced Force Vector (by eliminating the rows corresponding to support reactions):*

$$\{F^*\} = \begin{Bmatrix} 0 \\ 40 \end{Bmatrix}$$

*The reduced system is thus:*

$$\begin{Bmatrix} 0 \\ 40 \end{Bmatrix} = \mathbf{10^5} \begin{bmatrix} \mathbf{1.6814} & \mathbf{-0.0166} \\ \mathbf{-0.0166} & \mathbf{0.2765} \end{bmatrix} \begin{Bmatrix} u_2 \\ \theta_2 \end{Bmatrix}$$

*Which gives:* $\begin{Bmatrix} u_2 \\ \theta_2 \end{Bmatrix} = 10^{-5} \begin{Bmatrix} 1.428 \\ 145 \end{Bmatrix}$

***Forces in the Elements:***

*Element 1:*

$$\{D\}^1 = 10^{-5}\lfloor 0 \quad 0 \quad 0 \quad 1.428 \quad 0 \quad 145 \rfloor^T$$

$$\{d\}^1 = [T]^1\{D\}^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^{-5} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1.428 \\ 0 \\ 145 \end{Bmatrix} = 10^{-5} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.428 \\ 145 \end{Bmatrix}$$

$$\{f_n^1\} = \{f^1\} - \{f_{eq}^1\}$$

$$\{f_n^1\} = \frac{E}{L} \begin{bmatrix} \boldsymbol{A} & \mathbf{0} & \mathbf{0} & \boldsymbol{-A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{12}\boldsymbol{I}}{\boldsymbol{L^2}} & \frac{\mathbf{-6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{0} & \frac{\mathbf{-12}\boldsymbol{I}}{\boldsymbol{L^2}} & \frac{\mathbf{-6}\boldsymbol{I}}{\boldsymbol{L}} \\ \mathbf{0} & \frac{\mathbf{-6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{4}\boldsymbol{I} & \mathbf{0} & \frac{\mathbf{6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{2}\boldsymbol{I} \\ \boldsymbol{-A} & \mathbf{0} & \mathbf{0} & \boldsymbol{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{-12}\boldsymbol{I}}{\boldsymbol{L^2}} & \frac{\mathbf{6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{0} & \frac{\mathbf{12}\boldsymbol{I}}{\boldsymbol{L^2}} & \frac{\mathbf{6}\boldsymbol{I}}{\boldsymbol{L}} \\ \mathbf{0} & \frac{\mathbf{-6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{2}\boldsymbol{I} & \mathbf{0} & \frac{\mathbf{6}\boldsymbol{I}}{\boldsymbol{L}} & \mathbf{4}\boldsymbol{I} \end{bmatrix} \times 10^{-5} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.428 \\ 145 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f_n^1\} = 10^{-5}\frac{E}{L} \begin{Bmatrix} 0 \\ \frac{-12\boldsymbol{I}}{\boldsymbol{L^2}} \times -1.428 + \frac{-6\boldsymbol{I}}{\boldsymbol{L}} \times 145 \\ \frac{6\boldsymbol{I}}{\boldsymbol{L}} \times -1.428 + 2I \times 145 \\ 0 \\ \frac{12\boldsymbol{I}}{\boldsymbol{L^2}} \times -1.428 + \frac{6\boldsymbol{I}}{\boldsymbol{L}} \times 145 \\ \frac{6\boldsymbol{I}}{\boldsymbol{L}} \times -1.428 + 4I \times 145 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5.98 \\ 9.95 \\ 0 \\ 5.98 \\ 19.95 \end{Bmatrix}$$

*Element 2:*

$$\{D\}^2 = 10^{-5}\lfloor 1.428 \quad 0 \quad 145 \quad 0 \quad 0 \quad 0\rfloor^T$$

$$\{d\}^2 = [T]^2\{D\}^2 = = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^{-5} \begin{Bmatrix} 1.428 \\ 0 \\ 145 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 10^{-5} \begin{Bmatrix} 1.143 \\ -0.857 \\ 145 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f_n\}^2 = \{f\}^2 - \{f_{eq}\}^2$$

$$\{f_n^2\} = \frac{E}{L} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{-\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{0} & \frac{-\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{-\boldsymbol{6I}}{\boldsymbol{L}} \\ \boldsymbol{0} & \frac{-\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{4I} & \boldsymbol{0} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{2I} \\ -\boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{-\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{0} & \frac{\boldsymbol{12I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} \\ \boldsymbol{0} & \frac{-\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{2I} & \boldsymbol{0} & \frac{\boldsymbol{6I}}{\boldsymbol{L}} & \boldsymbol{4I} \end{bmatrix} \times 10^{-5} \begin{Bmatrix} 1.143 \\ -0.857 \\ 145 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -9 \\ -12 \\ 15 \\ -9 \\ -12 \\ -15 \end{Bmatrix} = \begin{Bmatrix} 11.96 \\ 5.98 \\ 5.06 \\ 6.04 \\ 18.02 \\ 25.04 \end{Bmatrix}$$

### *3.4.2 Hinged 2D Frame element:*

#### 3.4.2.1 Left hinged 2D Frame element:

$$\boldsymbol{u} = \left\lfloor \boldsymbol{1} - \frac{x}{L} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \frac{x}{L} \quad \boldsymbol{0} \quad \boldsymbol{0} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{v} = [N] \times \{\boldsymbol{d}\} = \left\lfloor 0 \quad \boldsymbol{1} - \frac{\boldsymbol{3x}}{\boldsymbol{2L}} + \frac{\boldsymbol{x^3}}{\boldsymbol{2L^3}} \quad 0 \quad 0 \quad \frac{\boldsymbol{3x}}{\boldsymbol{2L}} - \frac{\boldsymbol{x^3}}{\boldsymbol{2L^3}} \quad -\frac{\boldsymbol{x}}{\boldsymbol{2}} + \frac{\boldsymbol{x^3}}{\boldsymbol{2L^2}} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{\theta} = \frac{\boldsymbol{dv}}{\boldsymbol{dx}} = [N'] \times \{\boldsymbol{d}\} = \left\lfloor \boldsymbol{0} \quad -\frac{\boldsymbol{3}}{\boldsymbol{2L}} + \frac{\boldsymbol{3x^2}}{\boldsymbol{2L^3}} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \frac{\boldsymbol{3}}{\boldsymbol{2L}} - \frac{\boldsymbol{3x^2}}{\boldsymbol{2L^3}} \quad -\frac{\boldsymbol{1}}{\boldsymbol{2}} + \frac{\boldsymbol{3x^2}}{\boldsymbol{2L^2}} \right\rfloor \times \{\boldsymbol{d}\}$$

$$[\boldsymbol{k}] = \frac{\boldsymbol{E}}{\boldsymbol{L}} \begin{bmatrix} \boldsymbol{A} & 0 & 0 & -\boldsymbol{A} & 0 & 0 \\ 0 & \frac{\boldsymbol{3I}}{\boldsymbol{L^2}} & 0 & 0 & \frac{\boldsymbol{3I}}{\boldsymbol{L^2}} & \frac{\boldsymbol{3I}}{\boldsymbol{L}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\boldsymbol{A} & 0 & 0 & \boldsymbol{A} & 0 & 0 \\ 0 & \frac{-\boldsymbol{3I}}{\boldsymbol{L^2}} & 0 & 0 & \frac{\boldsymbol{3I}}{\boldsymbol{L^2}} & \frac{-\boldsymbol{3I}}{\boldsymbol{L}} \\ 0 & \frac{\boldsymbol{3I}}{\boldsymbol{L}} & 0 & 0 & \frac{-\boldsymbol{3I}}{\boldsymbol{L}} & 3\boldsymbol{I} \end{bmatrix}$$

$$[\mathbf{K}] = \frac{\mathbf{E}}{\mathbf{L}} \begin{bmatrix} +\left(\mathbf{A}l^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}\mathbf{m}^2\right) & +l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & 0 & -\left(\mathbf{A}l^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}\mathbf{m}^2\right) & -l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & -\frac{\mathbf{3mI}}{\mathbf{L}} \\ +l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & +\left(\mathbf{A}\mathbf{m}^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}l^2\right) & 0 & -l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & -\left(\mathbf{A}\mathbf{m}^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}l^2\right) & \frac{\mathbf{3}l\mathbf{I}}{\mathbf{L}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\left(\mathbf{A}l^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}\mathbf{m}^2\right) & -l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & 0 & +\left(\mathbf{A}l^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}\mathbf{m}^2\right) & +l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & \frac{\mathbf{3mI}}{\mathbf{L}} \\ -l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & -\left(\mathbf{A}\mathbf{m}^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}l^2\right) & 0 & +l\mathbf{m}\left(\mathbf{A} - \frac{\mathbf{3I}}{\mathbf{L^2}}\right) & +\left(\mathbf{A}\mathbf{m}^2 + \frac{\mathbf{3I}}{\mathbf{L^2}}l^2\right) & -\frac{\mathbf{3}l\mathbf{I}}{\mathbf{L}} \\ -\frac{\mathbf{3mI}}{\mathbf{L}} & \frac{\mathbf{3}l\mathbf{I}}{\mathbf{L}} & 0 & \frac{\mathbf{3mI}}{\mathbf{L}} & -\frac{\mathbf{3}l\mathbf{I}}{\mathbf{L}} & \mathbf{3I} \end{bmatrix}$$

**3.4.2.2 Right hinged 2D Frame element:**

$$u = \left\lfloor 1-\frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \quad 0 \right\rfloor \times \{d\}$$

$$v = [N] \times \{d\} = \left\lfloor 0 \quad 1-\frac{3x^2}{2L^2}+\frac{x^3}{2L^3} \quad x-\frac{3x^2}{2L}+\frac{x^3}{2L^2} \quad 0 \quad \frac{3x^2}{2L^2}-\frac{x^3}{2L^3} \quad 0 \right\rfloor \times \{d\}$$

$$\theta = \frac{dv}{dx} = [N'] \times \{d\}$$

$$= \left\lfloor 0 \quad -\frac{3x}{L^2}+\frac{3x^2}{2L^3} \quad 1-\frac{3x}{L}+\frac{3x^2}{2L^2} \quad 0 \quad \frac{3x}{L^2}-\frac{3x^2}{2L^3} \quad 0 \right\rfloor \times \{d\}$$

$$[k] = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{3I}{L^2} & \frac{3I}{L} & 0 & \frac{-3I}{L^2} & 0 \\ 0 & \frac{3I}{L} & 3I & 0 & \frac{-3I}{L} & 0 \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{-3I}{L^2} & \frac{-3I}{L} & 0 & \frac{3I}{L^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K] = \frac{E}{L}\begin{bmatrix} +\left(Al^2+\frac{3I}{L^2}m^2\right) & +lm\left(A-\frac{3I}{L^2}\right) & -\frac{3mI}{L} & -\left(Al^2+\frac{3I}{L^2}m^2\right) & -lm\left(A-\frac{3I}{L^2}\right) & 0 \\ +lm\left(A-\frac{3I}{L^2}\right) & +\left(Am^2+\frac{3I}{L^2}l^2\right) & \frac{3lI}{L} & -lm\left(A-\frac{3I}{L^2}\right) & -\left(Am^2+\frac{3I}{L^2}l^2\right) & 0 \\ -\frac{3mI}{L} & \frac{3lI}{L} & 3I & \frac{3mI}{L} & -\frac{3lI}{L} & 0 \\ -\left(Al^2+\frac{3I}{L^2}m^2\right) & -lm\left(A-\frac{3I}{L^2}\right) & \frac{3mI}{L} & +\left(Al^2+\frac{3I}{L^2}m^2\right) & +lm\left(A-\frac{3I}{L^2}\right) & 0 \\ -lm\left(A-\frac{3I}{L^2}\right) & -\left(Am^2+\frac{3I}{L^2}l^2\right) & -\frac{3lI}{L} & +lm\left(A-\frac{3I}{L^2}\right) & +\left(Am^2+\frac{3I}{L^2}l^2\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**3.4.2.3 Hinged 2D Frame element at its two nodes:**

$$u = \left\lfloor 1-\frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \quad 0 \right\rfloor \times \{d\}$$

$$v = [N] \times \{d\} = \left\lfloor 0 \quad 1-\frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \right\rfloor \times \{d\}$$

$$\theta = \frac{dv}{dx} = [N'] \times \{d\} = \left\lfloor 0 \quad -\frac{1}{L} \quad 0 \quad 0 \quad \frac{1}{L} \quad 0 \right\rfloor \times \{d\}$$

$$[\boldsymbol{k}] = \frac{\boldsymbol{E}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{A} & 0 & 0 & -\boldsymbol{A} & 0 & 0 \\ 0 & \dfrac{3\boldsymbol{I}}{\boldsymbol{L}^2} & 0 & 0 & \dfrac{-3\boldsymbol{I}}{\boldsymbol{L}^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\boldsymbol{A} & 0 & 0 & \boldsymbol{A} & 0 & 0 \\ 0 & \dfrac{-3\boldsymbol{I}}{\boldsymbol{L}^2} & 0 & 0 & \dfrac{3\boldsymbol{I}}{\boldsymbol{L}^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}] = \frac{\mathbf{E}}{\mathbf{L}}\begin{bmatrix} +\left(\mathbf{A}\boldsymbol{l}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\mathbf{m}^2\right) & +\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & 0 & -\left(\mathbf{A}\boldsymbol{l}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\mathbf{m}^2\right) & -\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & 0 \\ +\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & +\left(\mathbf{A}\mathbf{m}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\boldsymbol{l}^2\right) & 0 & -\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & -\left(\mathbf{A}\mathbf{m}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\boldsymbol{l}^2\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\left(\mathbf{A}\boldsymbol{l}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\mathbf{m}^2\right) & -\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & 0 & +\left(\mathbf{A}\boldsymbol{l}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\mathbf{m}^2\right) & +\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & 0 \\ -\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & -\left(\mathbf{A}\mathbf{m}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\boldsymbol{l}^2\right) & 0 & +\boldsymbol{l}\mathbf{m}\left(\mathbf{A} - \frac{3\mathbf{I}}{\mathbf{L}^2}\right) & +\left(\mathbf{A}\mathbf{m}^2 + \frac{3\mathbf{I}}{\mathbf{L}^2}\boldsymbol{l}^2\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$