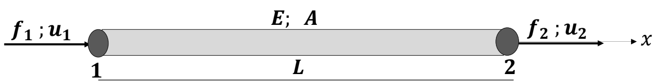
2.2. Tension-compression bar element and truss system

2.2.1 Definition of the tension-compression bar element, notations and sign conventions



2.2.2 Basic equations of the tension-compression bar element:

The deformation field is reduced to: $\boldsymbol{\varepsilon_x = \frac{\partial u}{\partial x} = \frac{du}{dx}}$

The stress field is reduced to: $\boldsymbol{\sigma_x = E\,\varepsilon_x}$

The normal force acting in a section of the bar is:

$$\boldsymbol{f = \int_A \sigma_x\, ds = \int_A E\varepsilon_x ds} = \mathrm{A}\boldsymbol{E\frac{du}{dx}}$$

The local equilibrium is therefore described by a single equation:

$$\boldsymbol{\frac{\partial \sigma_x}{\partial x} + f_x = 0}$$

The linear load $q(x)$ *(lateral friction) is transformed into volumetric load by division on the section A of the bar:*

$$\frac{d\sigma_x}{dx} + \frac{q}{A} = 0 \;\rightarrow$$

$$\frac{d}{dx}(E\varepsilon_x) + \frac{q}{A} = 0 \quad \rightarrow \frac{d}{dx}\left(\frac{Edu}{dx}\right) + \frac{q}{A} = 0$$

$$\boldsymbol{EA\frac{d^2u}{dx^2} + q = 0}$$

2.2.3. *Approximation of the displacement field. (Shape function):*

$$\boldsymbol{U = [N]\{d\}}$$

$$\boldsymbol{u = \left\lfloor 1 - \frac{x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}}$$

2.2.4. Define of the $deformation\ field$ ε as a function of the degrees of freedom

The deformation field $\boldsymbol{\varepsilon}$ is a differentiation of the displacement field:

$$\boldsymbol{\varepsilon = diff(U) = diff([N]\{d\}) = [B]\{d\}}$$

For the compression tension bar element:

$$\varepsilon_x = \frac{du}{dx} = \frac{d}{dx}\left(\left\lfloor 1-\frac{x}{L} \quad \frac{x}{L}\right\rfloor\right)\{d\} = \left\lfloor -\frac{1}{L} \quad \frac{1}{L}\right\rfloor \{d\}$$

$$\rightarrow [B] = \left\lfloor -\frac{1}{L} \quad \frac{1}{L}\right\rfloor$$

2.2.5. Define of $\boldsymbol{the\ stress\ field}$ $\boldsymbol{\sigma}$ as a function of the degrees of freedom:

The stress field $\boldsymbol{\sigma}$ is linked to the deformation field by the behavior law:

$$\boldsymbol{\sigma = [C]\varepsilon = [C][B]\{d\}}$$

For the compression tension bar element:

$$\boldsymbol{\sigma_x = E\ \varepsilon_x = E\left\lfloor -\frac{1}{L} \quad \frac{1}{L}\right\rfloor\{d\}}$$

So, the matrix $[\boldsymbol{C}]$ is reduced to the only Young modulus $\boldsymbol{E}$

2.2.6. Formulation of the element:

Formulation by the direct method:

For the compression tension bar element: $\boldsymbol{f} = \mathrm{A}\boldsymbol{E}\dfrac{\boldsymbol{du}}{\boldsymbol{dx}}$

The normal of the sections at the ends of the bar are -1 et 1

$$f_1 = -EA\left\lfloor -\frac{1}{L} \quad \frac{1}{L}\right\rfloor\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix};\quad f_2 = EA\left\lfloor -\frac{1}{L} \quad \frac{1}{L}\right\rfloor\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$\{\boldsymbol{f}\} = [\boldsymbol{k}]\{\boldsymbol{d}\};\ [\boldsymbol{k}] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix}$$

Formulation by the virtual work principle

$$\boldsymbol{W^{ext} = W^{int}}$$

In the case where we only have nodal charges: $\boldsymbol{W^{ext}} = \{\boldsymbol{d}^*\}^T.\{\boldsymbol{f}\}$

$$\boldsymbol{W^{int}} = \int_V \boldsymbol{\varepsilon^{*T}\,\sigma.dv} = \int_V ([\boldsymbol{B}]\{\boldsymbol{d}^*\})^T([\boldsymbol{C}][\boldsymbol{B}]\{\boldsymbol{d}\})\boldsymbol{dv}$$

$$\boldsymbol{W^{int}} = \int_V \{\boldsymbol{d}^*\}^T[\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}]\{\boldsymbol{d}\}\boldsymbol{dv} = \{\boldsymbol{d}^*\}^T\left(\int_V [\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}]\,\boldsymbol{dv}\right)\{\boldsymbol{d}\}$$

$$\{\boldsymbol{f}\} = [\boldsymbol{k}]\{\boldsymbol{d}\} \quad avec\ [\boldsymbol{k}] = \int_V [\boldsymbol{B}]^T[\boldsymbol{C}][\boldsymbol{B}]\,\boldsymbol{dv}$$

For the compression tension bar element:

$$[k] = \int_V \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} . E . \left[-\frac{1}{L} \quad \frac{1}{L} \right] dv =$$

For a bar of constant section A $V = AL$ from where

$$[k] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2.2.7. Treatment of distributed loads (or between nodes):

In the case where, in addition to the nodal loads, there are loads distributed within the volume of the element or on its external surface, the work of the external forces can be written as the sum of the work of the nodal forces and that of the work of the distributed forces.

$$W^{ext} = W^n + W^r = \{d^*\}^T . \{f_n\} + W^r$$

For the case of the tension compression bar, the bar can be subjected to a volume load f directed along its longitudinal axis and/or to concentrated friction $\boldsymbol{P_i}$ *at points with coordinates* $\boldsymbol{x_i}$ *or distributed over an interval* $[a \;\; b]$ *of its length*

$$W^r = \sum (u^*_{xi})^T P_i + \int_a^b (u^*(x))^T q(x) dx + \int_v (u^*(x))^T f dv$$

Or: $u^*(x) = [N(x)]\{d^*\}; \rightarrow (u^*(x))^T = \{d^*\}^T [N(x)]^T \;; \; (u^*_{xi})^T = \{d^*\}^T [N(x = xi)]^T$

$$W^r = \sum \{d^*\}^T [N(x = xi)]^T P_i + \int_a^b \{d^*\}^T [N(x)]^T q(x) dx + \int_v \{d^*\}^T [N(x)]^T f dv$$

$$W^r = \{d^*\}^T \left(\sum [N(x = xi)]^T P_i + \int_a^b [N(x)]^T q(x) dx + \int_v [N(x)]^T f dv \right)$$

$$W^r = \{d^*\}^T \{f_{eq}\}$$

$$\{f_{eq}\} = \sum [N(x = xi)]^T P_i + \int_a^b [N(x)]^T q(x) dx + \int_v [N(x)]^T f dv$$

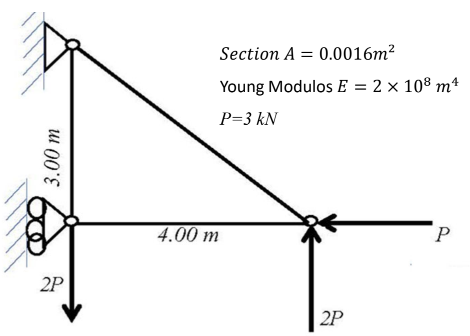
$$W^{ext} = \{d^*\}^T.\{f_n\} + \{d^*\}^T = \{d^*\}^T(\{f_n\} + \{f_{eq}\}) = \{d^*\}^T\{f\}$$

où $\{f\} = (\{f_n\} + \{f_{eq}\})$

Example: case of a compression tension bar loaded by friction varying linearly from 0 to k: $q = k\frac{x}{L}$

$$\{f_{eq}\} = \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} .k\frac{x}{L}.dx = k.\begin{bmatrix} \frac{x^2}{2L} - \frac{x^3}{3L^2} \\ \frac{x^3}{3L^2} \end{bmatrix}_0^L = \begin{bmatrix} \frac{kL}{6} \\ \frac{kL}{3} \end{bmatrix}$$

2.2.8. Study of plan truss

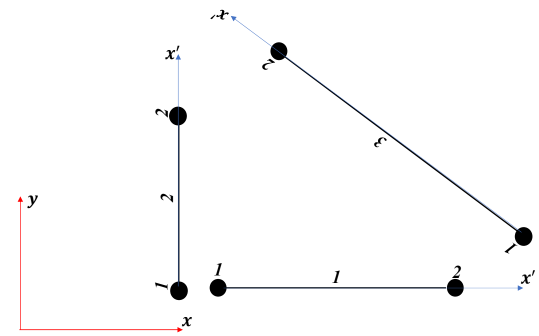


The transformation matrix is identical to that of the spring element, so the stiffness matrix of the compression tension bar element in the global frame is: $[K] = [T]^T[k][T]$

$$[K] = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[K] = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

For the truss above



Element 1 (bar1):

$L = 4 \, ; (\widehat{x' x}) = 0 \, ; \, (\widehat{x' y}) = 90 \, ; \;\; l = cos(0) = 1 \, ; \, m = \, cos(90) = 0$

$$[k]^1 = EA.\begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} \;\; \rightarrow$$

$$[K]^1 = EA\begin{bmatrix} 0.25 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \\ -0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2 (bar 2)

$L = 3 \, ; (\widehat{x' x}) = 90 \, ; \, (\widehat{x' y}) = 0 \, ; \;\; l = cos(0) = 0 \, ; \, m = \, cos(90) = 1$

$$[k]^2 = EA.\begin{bmatrix} 0.334 & -0.334 \\ -0.334 & 0.334 \end{bmatrix} \;\; \rightarrow$$

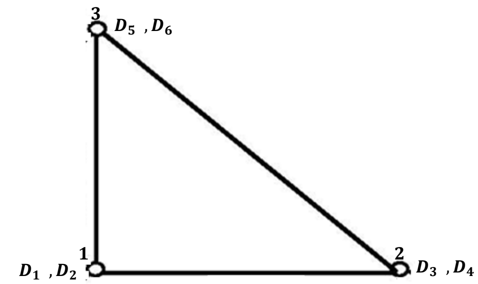
$$[K]^2 = EA\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.334 & 0 & -0.334 \\ 0 & 0 & 0 & 0 \\ 0 & -0.334 & 0 & 0.334 \end{bmatrix}$$

<u>*Element 3 (bar 3)*</u>

L=5 ; $l = \cos(\widehat{x' x}) = -\frac{4}{5} = -0.8$; $m = \cos(\widehat{x' y}) = \frac{3}{5} = 0.6$

$$[k]^3 = EA.\begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \rightarrow$$

$$[K]^3 = EA\begin{bmatrix} 0.128 & -0.096 & -0.128 & 0.096 \\ -0.096 & 0.072 & 0.096 & -0.072 \\ -0.128 & 0.096 & 0.128 & -0.096 \\ 0.096 & -0.072 & -0.096 & 0.072 \end{bmatrix}$$



For the truss in question, we have a total of 6 degrees of freedom, so we foresee a 6 x 6 matrix. The continuity of displacements is expressed by the following relationships

$$\{D\}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}; \qquad \{D\}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_5 \\ D_6 \end{Bmatrix};$$

$$\{D\}^3 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^3 = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

These allow the creation of the table below:

$[K] = EA \times$

0.25 0	0 0	-0.25	0	 0	 0
0 0	0 0.333	0	0	 0	 -0.333
-0.25	0	0.25 0.128	0 -0.096	 -0.128	 0.096
0	0	0 -0.096	0 0.072	 0.096	 -0.072
0	0	 -0.128	 0.096	0 0.128	0 -0.096
0	-0.333	 0.096	 -0.072	0 -0.096	0.333 0.072

The global system will therefore be:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} == EA \begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ 0 & 0.333 & 0 & 0 & 0 & -0.333 \\ -0.25 & 0 & 0.378 & -0.096 & -0.128 & 0.096 \\ 0 & 0 & -0.096 & 0.072 & 0.096 & -0.072 \\ 0 & 0 & -0.128 & 0.096 & 0.128 & -0.096 \\ 0 & -0.333 & 0.096 & -0.0072 & -0.096 & 0.405 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

Introduction of boundary conditions:

Node 1 : $F_1 = R_{x1}$; $F_2 = -2P$; $D_1 = 0$; : $D_2 = ?$

Node 2 :: $F_3 = -P$; $F_4 = 2P$; $D_3 = ?$; : $D_4 = ?$

Node 3 : $F_5 = R_{x3}$; $F_6 = R_{y3}$; $D_5 = 0$; : $D_6 = 0$

We take boundary conditions into account by eliminating the columns corresponding to the blocked degrees of freedom and the rows corresponding to the support reactions.

We then obtain the reduced system $\{F\}^* = [K]^*\{D\}^*$ *suivant :*

$$\begin{Bmatrix} -2P \\ -P \\ 2P \end{Bmatrix} = EA \begin{bmatrix} 0.333 & 0 & 0 \\ 0 & 0.378 & -0.096 \\ 0 & -0.096 & 0.072 \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Which gives:

$$\begin{Bmatrix} D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \frac{P}{EA} \begin{Bmatrix} -6.0060 \\ 6.6667 \\ 36.6667 \end{Bmatrix} = \begin{Bmatrix} -0.56250 \\ 0.6250 \\ 3.4375 \end{Bmatrix} \times 10^{-4}\, m$$

Determination of forces in the elements:

For each element: $\{f\} = [k]\{d\} - \{f_{eq}\}$; *Her* $\{f_{eq}\} = \boldsymbol{0}$

Element 1 :

$$\{D\}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.56250 \\ 0.6250 \\ 3.4375 \end{Bmatrix} \times 10^{-4}\, m;$$

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.5625 \\ 0.6250 \\ 3.4375 \end{Bmatrix} \times 10^{-4} = \begin{Bmatrix} 0 \\ 0.6250 \end{Bmatrix} \times 10^{-4}\, m$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^1 = EA \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix} \times 10^{-4} \begin{Bmatrix} 0 \\ 0.6250 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 5 \end{Bmatrix} kN$$

Element 2 :

$$\{D\}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.5625 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-4} \rightarrow$$

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.5625 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-4} = \begin{Bmatrix} -0.5625 \\ 0 \end{Bmatrix} \times 10^{-4}\, m$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^2 = EA \begin{bmatrix} 0.333 & -0.333 \\ -0.333 & 0.333 \end{bmatrix} \begin{Bmatrix} -0.5625 \\ 0 \end{Bmatrix} \times 10^{-4} = \begin{Bmatrix} -6 \\ 6 \end{Bmatrix} kN$$

Element 3 :

$$\{D\}^3 = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0.625 \\ 3.4375 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-4} \rightarrow$$

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^3 = \begin{bmatrix} -0.8 & 0.6 & 0 & 0 \\ 0 & 0 & -0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} 0.625 \\ 3.4375 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-4} = \begin{Bmatrix} 1.5625 \\ 0 \end{Bmatrix} \times 10^{-4}\, m$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^3 = EA. \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \begin{Bmatrix} 1.5625 \\ 0 \end{Bmatrix} \times 10^{-4} = \begin{Bmatrix} 10 \\ -10 \end{Bmatrix} kN$$

2.2.9. Spatial trusses (3D trusses):

The bar is in space, the direction cosines of the local axis are noted:

$$l_x = \cos(x'; x)\,;\; l_y = \cos(y'; y); \quad l_z = \cos(z'; z)\,,$$

or simply: *l, m, n*

The transformation matrix will be:

$$[T] = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

The stiffness matrix in the global coordinate will be:

$$[K] = \frac{EA}{L} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}$$