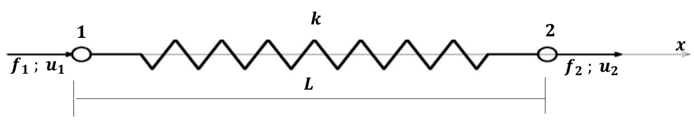
2.1. Spring element and springs system:

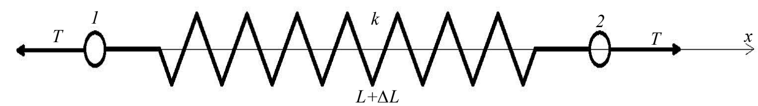
2.1.1. Definition of the spring element, notations and sign conventions

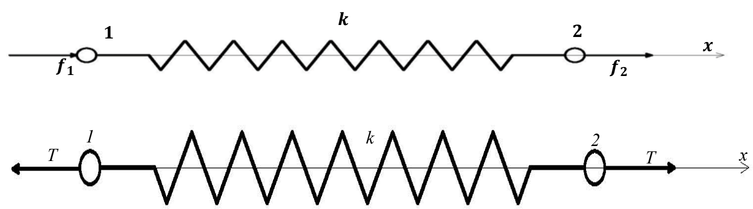


2.1.2. Element Formulation :

The formulation of this element can be done in a simple way by the direct equilibrium method.

Let's examine a spring subjected to a tension T at its ends.





By comparaison : $f_1 = -T$; $f_2 = T$

Which verifies the equilibrium.: $f_1 + f_2 = -T + T = 0$

We have: $T = k\Delta L = k(u_2 - u_1)$

$$\Delta L = (L + u_2 - u_1 - L) = u_2 - u_1$$

Which gives:

$$f_1 = -k(u_2 - u_1) = k(u_1 - u_2)$$

$$f_2 = k(u_2 - u_1) = k(-u_1 + u_2)$$

Or in matrix form:

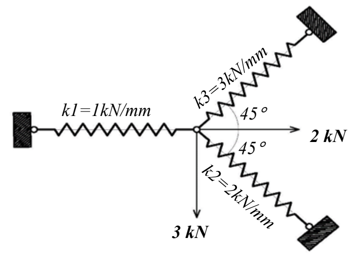
$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

The stiffness matrix of the spring element is:

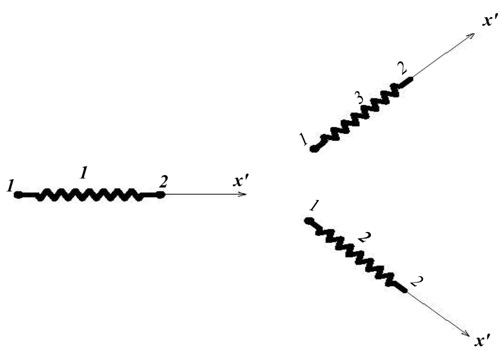
$$[k] = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2.1.3. Study of a spring system:

Let’s consider the spring system shown in the figure below.



2.1.3.1. Determination of the equation systems of the elements:



The separate study of each of the springs 1, 2, and 3 respectively gives:

$$\{f\}^1 = [k]^1\{u\}^1 = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^1 \rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^1 = 10^3 \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^1$$

$$\{f\}^2 = [k]^2\{u\}^2 = k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^2 \rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^1 = 10^3 \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^2$$

$$\{f\}^3 = [k]^3\{u\}^3 = k_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^2 \rightarrow \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^3 = 10^3 \times \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}^3$$

2.1.3.2. Determination of the global system of the problem:

Transformation of the different quantities from local coordinates to global coordinates:

If $[T]$ *(called the transformation matrix) is the matrix that converts local coordinates to global coordinates such that:*

$$\{d\} = [T]\{D\} \ et \ \{f\} = [T]\{F\}$$

Where : $\{d\}$ and $\{f\}$ *are the nodal displacements and nodal forces of the element in local coordinates, and* $\{D\}$ and $\{F\}$ *are the nodal displacements and nodal forces of the element in global coordinates.*

The system: $\{\boldsymbol{f}\} = [\boldsymbol{k}]\{\boldsymbol{d}\}$ *in local coordinates can be transformed as follows:*

$$\{f\} = [k]\{d\} \to [T]\{F\} = [k][T]\{D\}$$

$$[T]^{-1}[T]\{F\} = [T]^{-1}[k][T]\{D\} \to$$

$$\{F\} = ([T]^{-1}[k][T])\{D\}$$

$$\{F\} = [K]\{D\}$$

with

$$[K] = [T]^{-1}[k][T]$$

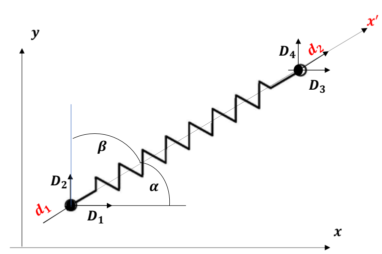
We will check that $[T]$ *is an orthogonal matrix:*

$$\to \ [T]^{-1} = [T]^{T};$$

The stiffness matrix of the element in global coordinates $[K]$ *is then derived from the stiffness matrix in local coordinates* $[k]$ *and the transformation matrix* $[T]$ *by :*

$$[\boldsymbol{K}] = [\boldsymbol{T}]^{\boldsymbol{T}}[\boldsymbol{k}][\boldsymbol{T}]$$

For a spring element having any direction in the plane of displacement:



Noting: $l_x = \cos(x', x)$ *and* $l_y = \cos(x', y)$

We will have:

$$\boldsymbol{d_1 = l_x D_1 + l_y D_2} \quad \text{and} \quad \boldsymbol{d_2 = l_x D_3 + l_y D_4}$$

In this case plan, to lighten the writing, we pose:

$l_x = l \quad et \quad l_y = m$; *which gives:*

$$[\boldsymbol{T}] = \begin{bmatrix} \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{l} & \boldsymbol{m} \end{bmatrix}$$

The stiffness matrix of the spring element in global coordinates will be:

$$[\boldsymbol{K}] = \begin{bmatrix} \boldsymbol{l} & \boldsymbol{0} \\ \boldsymbol{m} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{l} \\ \boldsymbol{0} & \boldsymbol{m} \end{bmatrix} \times \boldsymbol{k} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{l} & \boldsymbol{m} \end{bmatrix}$$

$$[\boldsymbol{K}] = \boldsymbol{k} \begin{bmatrix} \boldsymbol{l^2} & \boldsymbol{lm} & \boldsymbol{-l^2} & \boldsymbol{-lm} \\ \boldsymbol{lm} & \boldsymbol{m^2} & \boldsymbol{-lm} & \boldsymbol{-m^2} \\ \boldsymbol{-l^2} & \boldsymbol{-lm} & \boldsymbol{l^2} & \boldsymbol{lm} \\ \boldsymbol{-lm} & \boldsymbol{-m^2} & \boldsymbol{lm} & \boldsymbol{m^2} \end{bmatrix}$$

Spring 1 : $\boldsymbol{(x', x) = 0 \rightarrow l = 1, m = 0 \rightarrow}$

$$[\boldsymbol{T}]^{\boldsymbol{1}} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \end{bmatrix},$$

$$[\boldsymbol{K}]^{\boldsymbol{1}} = \boldsymbol{10^3} \times \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

Spring 2 : $\boldsymbol{(x', x) = -45° \rightarrow l = \frac{\sqrt{2}}{2}, m = -\frac{\sqrt{2}}{2} \rightarrow}$

$$[\boldsymbol{T}]^{\boldsymbol{2}} = \frac{\sqrt{2}}{2} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{-1} \end{bmatrix},$$

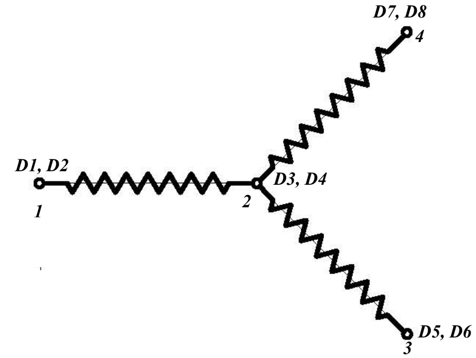
$$[\boldsymbol{K}]^{\boldsymbol{2}} == \boldsymbol{10^3} \times \begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{-1} & \boldsymbol{1} \\ \boldsymbol{-1} & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{-1} & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix}$$

Spring 3 : $(\boldsymbol{x}', \boldsymbol{x}) = \mathbf{45°} \rightarrow \boldsymbol{l} = \frac{\sqrt{2}}{2}, \boldsymbol{m} = \frac{\sqrt{2}}{2} \rightarrow$

$$[\boldsymbol{T}]^3 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$[\boldsymbol{K}]^3 == 10^3 \begin{bmatrix} 1.5 & 1.5 & -1.5 & -1.5 \\ 1.5 & 1.5 & -1.5 & -1.5 \\ -1.5 & -1.5 & 1.5 & 1.5 \\ -1.5 & -1.5 & 1.5 & 1.5 \end{bmatrix}$$

Assembly by the direct rigidity method:



Considering the continuity of displacements, we can write:

$$\{D\}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}; \{D\}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^2 = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}; \{D\}^3 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^3 = \begin{Bmatrix} D_3 \\ D_4 \\ D_7 \\ D_8 \end{Bmatrix}$$

For an element (k) of a structure composed of « m » elements governed in the local reference frame by the system:

$\{f\}^k = [k]^k\{d\}^k$ *and in the global reference frame by the system* $\{F\}^k = [K]^k\{D\}^k$*;*

Each F_i^k *is written as a function of all degrees of freedom of the entire structure:*

$$F_1^k = K_{11}^k D_1 + K_{12}^k D_2 + ...K_{1n}^k D_n$$
$$F_2^k = K_{21}^k D_1 + K_{22}^k D_2 + ...K_{2n}^k D_n$$
$$\vdots$$
$$F_n^k = K_{n1}^k D_1 + K_{n2}^k D_2 + ...K_{nn}^k D_n$$

If, for example, P_i is the external force applied in the direction of the degree of freedom « i » then the equilibrium gives:

$$P_i = F_i^1 + F_i^2 + ... + F_i^m$$

$$P_i = (K_{11}^1 + K_{11}^2 + ... + K_{11}^m)D_1 + (K_{12}^1 + K_{12}^2 + ... + K_{12}^m)D_2 + ... + (K_{1n}^1 + K_{1n}^2 + ... + K_{1n}^m)D_n$$

$$P_i = K_{11}D_1 + K_{12}D_2 + ... + K_{1n}D_n$$

With : $\boldsymbol{K_{ij}} = \boldsymbol{K_{ij}^1} + \boldsymbol{K_{ij}^2} + ... + \boldsymbol{K_{ij}^m} = \sum_{k=1}^{m} \boldsymbol{K_{ij}^k}$

From this analysis we can derive the following algorithm for the construction of the equations of the complete structure:

1. *For each stiffness coefficient $\boldsymbol{K_{p,q}^k}$ of an element « k » we determine its position (i , j) in the global matrix **K**, the first index designates the force for which the equation is written and the second designates the number of the degree of freedom considered.*
2. *We provide a table having the form of a square matrix whose order is equal to the total number of degrees of freedom of the structure, each of the elements of the table is identified by two indices i, j.*
3. *We place each stiffness coefficient of element 1 in the case corresponding to row i and column j.*

4. Operations 1 and 3 above are repeated for each element until all the elements are used up. If a coefficient is to be positioned in a location already occupied, it is the sum of the two values that will be placed, thus obtaining the complete system of stiffness coefficients

$[\mathbf{K}] = \mathbf{10^3} \times$

1	0	-1	0				
0	0	0	0				
-1	0	1 1 1.5	0 -1 1.5	-1	1	-1.5	-1.5
0	0	0 -1 1.5	0 1 1.5	1	-1	-1.5	-1.5
		-1	1	1	-1		
		1	-1	-1	1		
		-1.5	-1.5			1.5	1.5
		-1.5	-1.5			1.5	1.5

The global system is:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = 10^3 \times \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3.5 & 0.5 & -1 & 1 & -1.5 & -1.5 \\ 0 & 0 & 0.5 & 2.5 & 1 & -1 & -1.5 & -1.5 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1.5 & -1.5 & 0 & 0 & 1.5 & 1.5 \\ 0 & 0 & -1.5 & -1.5 & 0 & 0 & 1.5 & 1.5 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \end{Bmatrix}$$

2.1.3.3 Introduction of boundary conditions:

We have:

$F_1 = R_{x1}\,,\ F_2 = R_{y1}\,,\ F_3 = 2\ kN,\ \ F_4 = 3\ kN,\ \ F_5 = R_{x3}\,,$

$F_6 = R_{y3}\,,\ F_7 = R_{x4}\,,\ \ F_8 = R_{y4}$

$D_1 = D_2 = 0\,, D_3 = ?\ \ D_4 = ?\ \ , D_5 = D_6 = D_7 = D_8 = 0$

Note that in this system , the columns corresponding to the blocked degrees of freedom will be multiplied by 0 and that the corresponding rows introduce unknown support reactions, not principal in this calculation phase, we can therefore form the reduced system, noted: $\{F\}^* = [K]^*\{D\}^*$ *by eliminating these columns and these rows. (Columns corresponding to the blocked degrees of freedom and rows corresponding to the support reactions)*

$$\begin{Bmatrix} 2 \\ -3 \end{Bmatrix} = 10^3 \times \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix}$$

The resolution gives the displacements at nodes 2.

$$\begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix} = 10^{-4} \times \begin{Bmatrix} 7.647 \\ -13.529 \end{Bmatrix}$$

2.1.3.4. Determination of the forces at the nodes of the elements

To calculate the forces at the nodes of an element "i" (here spring), we deduce $\{\boldsymbol{D}\}^{\boldsymbol{i}}$ *from the degrees of freedom of the structure* $\{\boldsymbol{D}\}$ *using the continuity of the displacements, then we determine its degrees of freedom in local reference by the formula::* $\{\boldsymbol{d}\}^{\boldsymbol{i}} = [\boldsymbol{T}]^{\boldsymbol{i}}\{\boldsymbol{D}\}^{\boldsymbol{i}}$

Finally, we use the systems of elements in local reference $\{\boldsymbol{f}\}^{\boldsymbol{i}} = [\boldsymbol{k}]^{\boldsymbol{i}}\{\boldsymbol{d}\}^{\boldsymbol{i}}$ *to determine the nodal forces of the element, we will therefore have:*

Spring 1 : $\{\boldsymbol{D}\}^{1} = \begin{Bmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \\ \boldsymbol{D_3} \\ \boldsymbol{D_4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 7.647 \\ -13.53 \end{Bmatrix} 10^{-4} \rightarrow \{\boldsymbol{d}\}^{1} = [\mathbf{T}]^{1}\{D\}^{1}$

$$\{\boldsymbol{d}\}^{1} = \begin{Bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{Bmatrix}^{1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 7.647 \\ -13.529 \end{Bmatrix} 10^{-4} = \begin{Bmatrix} 0 \\ 0.000765 \end{Bmatrix} m$$

$$\{\boldsymbol{f}\}^{1} = [\boldsymbol{k}]^{1}\{\boldsymbol{d}\}^{1} \rightarrow \begin{Bmatrix} \boldsymbol{f_1} \\ \boldsymbol{f_2} \end{Bmatrix}^{1} = \mathbf{10^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.000765 \end{Bmatrix} = \begin{Bmatrix} -0.765 \\ 0.765 \end{Bmatrix} kN$$

Spring 2 : $\{\boldsymbol{D}\}^{2} = \begin{Bmatrix} \boldsymbol{D_3} \\ \boldsymbol{D_4} \\ \boldsymbol{D_5} \\ \boldsymbol{D_6} \end{Bmatrix} = \begin{Bmatrix} 7.65 \\ -13.53 \\ 0 \\ 0 \end{Bmatrix} 10^{-4} \rightarrow \{\boldsymbol{d}\}^{2} = [\mathbf{T}]^{2}\{D\}^{2} =$

$$\{\boldsymbol{d}\}^{2} = \begin{Bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{Bmatrix}^{2} = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} 7.65 \\ -13.53 \\ 0 \\ 0 \end{Bmatrix} 10^{-4} = \begin{Bmatrix} 0.001497 \\ 0 \end{Bmatrix} m$$

$$\{\boldsymbol{f}\}^{2} = [\boldsymbol{k}]^{2}\{\boldsymbol{d}\}^{2} \rightarrow \begin{Bmatrix} \boldsymbol{f_1} \\ \boldsymbol{f_2} \end{Bmatrix}^{2} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \mathbf{10^3} \begin{Bmatrix} 0.001497 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -3 \end{Bmatrix} kN$$

Spring 3 : $\{\boldsymbol{D}\}^{3} = \begin{Bmatrix} \boldsymbol{D_3} \\ \boldsymbol{D_4} \\ \boldsymbol{D_7} \\ \boldsymbol{D_8} \end{Bmatrix} = \begin{Bmatrix} 7.647 \\ -13.53 \\ 0 \\ 0 \end{Bmatrix} 10^{-4} \rightarrow \{\boldsymbol{d}\}^{3} = [\mathbf{T}]^{3}\{D\}^{3}$

$$\{\boldsymbol{d}\}^{3} = \begin{Bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \end{Bmatrix}^{3} = \frac{\sqrt{\mathbf{2}}}{\mathbf{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 7.647 \\ -13.53 \\ 0 \\ 0 \end{Bmatrix} 10^{-4} = \begin{Bmatrix} -0.000416 \\ 0 \end{Bmatrix} m$$

$$\{\boldsymbol{f}\}^{3} = [\boldsymbol{k}]^{3}\{\boldsymbol{d}\}^{3} \rightarrow \begin{Bmatrix} \boldsymbol{f_1} \\ \boldsymbol{f_2} \end{Bmatrix}^{3} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{10^3} \begin{Bmatrix} -0.000416 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ 1.25 \end{Bmatrix} kN$$