

FILIERE : Génie Civil

SPECIALITE : Master Géotechnique (M1/S2)

COURS : Rhéologie des sols

INTERROGATION N° : 1

(Date : 13/04/2026 - Durée : 1 heure 30)

A. (12 pts.) Soit le champ de déplacement défini en un point d'un solide élastique isotrope :

$$\bar{u} \begin{cases} u_1 = k(x_2 - x_3) \\ u_2 = k(x_1 + x_3) \\ u_3 = k(x_1 + x_2) \end{cases} / (\bar{x}_1, \bar{x}_2, \bar{x}_3),$$

où $k \geq 0$ désigne une constante positive petite pour assurer la linéarité géométrique.

Construire le tenseur des déformations $\bar{\varepsilon} = [\varepsilon_{ij}]$ correspondant, puis déterminer le tenseur des contraintes $\bar{\sigma} = [\sigma_{ij}]$ résultant de la sollicitation appliquée.

B. (8 pts.) Soit le tenseur des contraintes appliqué au même point du solide élastique isotrope considéré ci-dessus :

$$\bar{\sigma} = [\sigma_{ij}] = 2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad (N/mm^2),$$

où G désigne le module de cisaillement du matériau.

Construire le tenseur des déformations $\bar{\varepsilon} = [\varepsilon_{ij}]$ résultant de la sollicitation appliquée, puis déduire la (ou les) relation (s) possible (s) entre les paramètres de comportement utilisés ($\lambda, \mu, E, \nu, K, G$).

N.B. Seules les notes de cours sont autorisées.

Bonne chance

Prof. Mohamed Khemissa

Partie A

A1. Tenseur des déformations

Il est défini comme suit :

$$\bar{\bar{\epsilon}} = (\epsilon_{ij}) = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \quad / (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

tel que :

$$\bar{\bar{\epsilon}} = \frac{1}{2} [\overline{\text{grad}}(\bar{u}) + {}^t\overline{\text{grad}}(\bar{u})] \quad \Leftrightarrow \quad \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

D'où :

- $\epsilon_{11} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) = \frac{\partial u_1}{\partial x_1} = 0$
- $\epsilon_{22} = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) = \frac{\partial u_2}{\partial x_2} = 0$
- $\epsilon_{33} = \frac{1}{2} \left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \right) = \frac{\partial u_3}{\partial x_3} = 0$
- $\epsilon_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (k + k) = k \quad (= \epsilon_{21})$
- $\epsilon_{13} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2} (-k + k) = 0 \quad (= \epsilon_{31})$
- $\epsilon_{23} = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) = \frac{1}{2} (k + k) = k \quad (= \epsilon_{32})$

$$\Rightarrow \bar{\bar{\epsilon}} = [\epsilon_{ij}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad (\#6)$$

Remarque : $Tr(\bar{\bar{\epsilon}}) = \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0 \Rightarrow$ Tenseur déviatorique

A2. Tenseur des contraintes

1^{ère} Solution

Sachant que :

$$\bar{\sigma} = \lambda Tr(\bar{\bar{\epsilon}}) \bar{\delta} + 2\mu \bar{\bar{\epsilon}} \quad \Leftrightarrow \quad \sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

telle que : $Tr(\bar{\bar{\epsilon}}) = \epsilon_{kk} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = 0 + 0 + 0 = 0$

D'où :

- $\sigma_{11} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{11} + 2\mu \times \epsilon_{11} = \lambda \times 0 \times 1 + 2\mu \times 0 = 0$
- $\sigma_{22} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{22} + 2\mu \times \epsilon_{22} = \lambda \times 0 \times 1 + 2\mu \times 0 = 0$
- $\sigma_{33} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{33} + 2\mu \times \epsilon_{33} = \lambda \times 0 \times 1 + 2\mu \times 0 = 0$
- $\sigma_{12} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{12} + 2\mu \times \epsilon_{12} = \lambda \times 0 \times 0 + 2\mu \times (k) = 2\mu k \quad (= \sigma_{21})$
- $\sigma_{13} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{13} + 2\mu \times \epsilon_{13} = \lambda \times 0 \times 0 + 2\mu \times (0) = 0 \quad (= \sigma_{31})$
- $\sigma_{23} = \lambda \times (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \times \delta_{23} + 2\mu \times \epsilon_{23} = \lambda \times 0 \times 0 + 2\mu \times (k) = 2\mu k \quad (= \sigma_{32})$

$$\Rightarrow \bar{\sigma} = [\sigma_{ij}] = 2\mu \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad (\#6) \Rightarrow \bar{\sigma} = 2\mu \bar{\varepsilon} \Leftrightarrow \sigma_{ij} = 2\mu \varepsilon_{ij}$$

Remarque 1 : $Tr(\bar{\sigma}) = \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0 \Rightarrow$ Tenseur déviatorique

Remarque 2 : Sachant que : $Tr(\bar{\varepsilon}) = \varepsilon_{kk} = 0 \Rightarrow \bar{\sigma} = 2\mu \bar{\varepsilon} \Leftrightarrow \sigma_{ij} = 2\mu \varepsilon_{ij}$

$$\Rightarrow \bar{\sigma} = [\sigma_{ij}] = 2\mu \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad Ok.$$

2ème Solution

Sachant que :

$$\bar{\sigma} = K Tr(\bar{\varepsilon}) \bar{\delta} + 2G \bar{e} \Leftrightarrow \sigma_{ij} = K \varepsilon_{kk} \delta_{ij} + 2G e_{ij}$$

$$\bar{e} = \bar{\varepsilon} - \frac{1}{3} Tr(\bar{\varepsilon}) \bar{\delta} \Leftrightarrow e_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij}$$

telle que : $Tr(\bar{\varepsilon}) = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 + 0 + 0 = 0$

D'où :

- $e_{11} = \varepsilon_{11} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{11} = 0 - \frac{1}{3}(0) \times 1 = 0$
- $e_{22} = \varepsilon_{22} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{22} = 0 - \frac{1}{3}(0) \times 1 = 0$
- $e_{33} = \varepsilon_{33} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{33} = 0 - \frac{1}{3}(0) \times 1 = 0$
- $e_{12} = \varepsilon_{12} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{12} = k - \frac{1}{3}(0) \times 0 = k \quad (= e_{21})$
- $e_{13} = \varepsilon_{13} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{13} = 0 - \frac{1}{3}(0) \times 0 = 0 \quad (= e_{31})$
- $e_{23} = \varepsilon_{23} - \frac{1}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{23} = k - \frac{1}{3}(0) \times 0 = k \quad (= e_{32})$

$$\Rightarrow \bar{e} = [e_{ij}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \Rightarrow \bar{e} = \bar{\varepsilon}$$

D'où :

- $\sigma_{11} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{11} + 2G \times e_{11} = K \times (0) \times 1 + 2G \times 0 = 0$
- $\sigma_{22} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{22} + 2G \times e_{22} = K \times (0) \times 1 + 2G \times 0 = 0$
- $\sigma_{33} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{33} + 2G \times e_{33} = K \times (0) \times 1 + 2G \times 0 = 0$
- $\sigma_{12} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{12} + 2G \times e_{12} = K \times (0) \times 0 + 2G \times k = 2Gk \quad (= \sigma_{21})$
- $\sigma_{13} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{13} + 2G \times e_{13} = K \times (0) \times 0 + 2G \times 0 = 0 \quad (= \sigma_{31})$
- $\sigma_{23} = K \times (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \times \delta_{23} + 2G \times e_{23} = K \times (0) \times 0 + 2G \times k = 2Gk \quad (= \sigma_{32})$

$$\Rightarrow \bar{\sigma} = [\sigma_{ij}] = 2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \Rightarrow \bar{\sigma} = 2G \bar{e} \Leftrightarrow \sigma_{ij} = 2G \varepsilon_{ij}$$

Remarque 1 : $Tr(\bar{\sigma}) = \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0 \Rightarrow$ Tenseur déviatorique

Remarque 2 : Sachant que : $Tr(\bar{\varepsilon}) = \varepsilon_{kk} = 0 \Rightarrow \bar{\varepsilon} = \bar{\bar{\varepsilon}} \Leftrightarrow e_{ij} = \varepsilon_{ij}$

$$\Rightarrow \bar{\sigma} = 2G\bar{\varepsilon} = 2G\bar{\bar{\varepsilon}} \Leftrightarrow \sigma_{ij} = 2Ge_{ij} = 2G\varepsilon_{ij}$$

$$\Rightarrow \bar{\sigma} = [\sigma_{ij}] = 2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad Ok$$

Partie B

B1. Tenseur des déformations

1^{ère} Solution

Sachant que :

$$\bar{\varepsilon} = -\frac{\nu}{E} Tr(\bar{\sigma}) \bar{\delta} + \frac{1+\nu}{E} \bar{\sigma} \Leftrightarrow \varepsilon_{ij} = -\frac{\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij}$$

telle que : $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0 + 0 + 0 = 0$

D'où :

- $\varepsilon_{11} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{11} + \frac{1+\nu}{E} \times \sigma_{11} = -\frac{\nu}{E} \times 0 \times 1 + \frac{1+\nu}{E} \times 0 = 0$
- $\varepsilon_{22} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{22} + \frac{1+\nu}{E} \times \sigma_{22} = -\frac{\nu}{E} \times 0 \times 1 + \frac{1+\nu}{E} \times 0 = 0$
- $\varepsilon_{33} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{33} + \frac{1+\nu}{E} \times \sigma_{33} = -\frac{\nu}{E} \times 0 \times 1 + \frac{1+\nu}{E} \times 0 = 0$
- $\varepsilon_{12} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{12} + \frac{1+\nu}{E} \times \sigma_{12} = -\frac{\nu}{E} \times 0 \times 0 + \frac{1+\nu}{E} \times k = \frac{1+\nu}{E} 2Gk \quad (= \sigma_{21})$
- $\varepsilon_{13} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{13} + \frac{1+\nu}{E} \times \sigma_{13} = -\frac{\nu}{E} \times 0 \times 0 + \frac{1+\nu}{E} \times 0 = 0 \quad (= \sigma_{31})$
- $\varepsilon_{23} = -\frac{\nu}{E} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{23} + \frac{1+\nu}{E} \times \sigma_{23} = -\frac{\nu}{E} \times 0 \times 0 + \frac{1+\nu}{E} \times k = \frac{1+\nu}{E} 2Gk \quad (= \sigma_{32})$

$$\Rightarrow \bar{\varepsilon} = [\varepsilon_{ij}] = \frac{1+\nu}{E} \left(2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad (\#6) \Rightarrow \bar{\varepsilon} = \frac{1+\nu}{E} \bar{\sigma} \Leftrightarrow \varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij}$$

Remarque 1 : $Tr(\varepsilon) = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \equiv 0 \Rightarrow$ Tenseur déviatorique

Remarque 2 : Sachant que : $Tr(\bar{\sigma}) = \sigma_{kk} = 0 \Rightarrow \bar{\varepsilon} = \frac{1+\nu}{E} \bar{\sigma} \Leftrightarrow \varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij}$

$$\Rightarrow \bar{\varepsilon} = [\varepsilon_{ij}] = \frac{1+\nu}{E} \left(2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad Ok$$

2^{ème} Solution :

Sachant que :

$$\bar{\varepsilon} = \frac{1}{3K} Tr(\bar{\sigma}) \bar{\delta} + \frac{1}{2G} \bar{s} \Leftrightarrow \varepsilon_{ij} = \frac{1}{3K} \sigma_{kk} \delta_{ij} + \frac{1}{2G} s_{ij}$$

$$\bar{s} = \bar{\sigma} - \frac{1}{3} Tr(\bar{\sigma}) \bar{\delta} \Leftrightarrow s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

telle que : $Tr(\bar{\sigma}) = \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 0 + 0 + 0 = 0$

D'où :

- $s_{11} = \sigma_{11} - \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{11} = 0 - \frac{1}{3} (0) \times 1 = 0$

- $s_{22} = \sigma_{22} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{22} = 0 - \frac{1}{3}(0) \times 1 = 0$
- $s_{33} = \sigma_{33} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{33} = 0 - \frac{1}{3}(0) \times 1 = 0$
- $s_{12} = \sigma_{12} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{12} = 2Gk - \frac{1}{3}(0) \times 0 = 2Gk \quad (= s_{21})$
- $s_{13} = \sigma_{13} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{13} = 0 - \frac{1}{3}(0) \times 0 = 0 \quad (= s_{31})$
- $s_{23} = \sigma_{23} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{23} = 2Gk - \frac{1}{3}(0) \times 0 = 2Gk \quad (= e_{32})$

$$\Rightarrow \bar{s} = [s_{ij}] = 2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \Rightarrow \bar{s} = \bar{\sigma}$$

D'où :

- $\varepsilon_{11} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{11} + \frac{1}{2G} \times s_{11} = \frac{1}{3K} \times (0) \times 1 + \frac{1}{2G} \times (0) = 0$
- $\varepsilon_{22} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{22} + \frac{1}{2G} \times s_{22} = \frac{1}{3K} \times (0) \times 1 + \frac{1}{2G} \times (0) = 0$
- $\varepsilon_{33} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{33} + \frac{1}{2G} \times s_{33} = \frac{1}{3K} \times (0) \times 1 + \frac{1}{2G} \times (0) = 0$
- $\varepsilon_{12} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{12} + \frac{1}{2G} \times s_{12} = \frac{1}{3K} \times (0) \times 0 + \frac{1}{2G} \times (2Gk) = k \quad (= \varepsilon_{21})$
- $\varepsilon_{13} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{13} + \frac{1}{2G} \times s_{13} = \frac{1}{3K} \times (0) \times 0 + \frac{1}{2G} \times (0) = 0 \quad (= \varepsilon_{31})$
- $\varepsilon_{23} = \frac{1}{3K} \times (\sigma_{11} + \sigma_{22} + \sigma_{33}) \times \delta_{23} + \frac{1}{2G} \times s_{23} = \frac{1}{3K} \times (0) \times 0 + \frac{1}{2G} \times (2Gk) = k \quad (= \varepsilon_{32})$

$$\Rightarrow \bar{\varepsilon} = [\varepsilon_{ij}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad Ok.$$

Remarque 1 : $Tr(\varepsilon) = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \equiv 0 \Rightarrow$ Tenseur déviatorique

Remarque 2 : Sachant que : $Tr(\bar{\sigma}) = \sigma_{kk} = 0 \Rightarrow \bar{s} = \bar{\sigma} \Leftrightarrow s_{ij} = \sigma_{ij}$
 $\Rightarrow \bar{\varepsilon} = \frac{1}{2G} \bar{s} = \frac{1}{2G} \bar{\sigma} \Leftrightarrow \varepsilon_{ij} = \frac{1}{2G} s_{ij} = \frac{1}{2G} \sigma_{ij}$

$$\Rightarrow \bar{\varepsilon} = [\varepsilon_{ij}] = \frac{1}{2G} \left(2G \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix} \quad Ok$$

B2. Paramètres de comportement

Les relations possibles entre les paramètres de comportement utilisés ci-dessus sont les suivantes :

- Sachant que : $\bar{\sigma} = 2\mu\bar{\varepsilon} \quad \& \quad \bar{\sigma} = 2G\bar{\varepsilon} \Rightarrow \mu = G \quad (\#1)$
- Sachant que : $\bar{\sigma} = 2G\bar{\varepsilon} \quad \& \quad \bar{\varepsilon} = \frac{1+\nu}{E} \bar{\sigma} \Rightarrow G = \frac{E}{2(1+\nu)} \quad (\#1)$