Recapitulative
Récapitulatif

1. Approximation of displacement fields of some elements:
(Approximation des champs de déplacements de quelques éléments)

1.1. Two-nodes tension–compression bar element:
(Elément barre de traction-compression à deux nœuds)

$$U = u \quad et \ \{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},$$

$$u = [N] \times \{d\} = \lfloor N_1 \quad N_2 \rfloor \{d\} = \left\lfloor 1 - \frac{x}{L} \quad \frac{x}{L} \right\rfloor \times \{d\}$$

1.2. Two-nodes torsion bar element:
(Elément barre de torsion à deux nœuds)

$$U = \theta_x \quad et \ \{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\theta_x = [N] \times \{d\} = \lfloor N_1 \quad N_2 \rfloor \{d\} = \left\lfloor 1 - \frac{x}{L} \quad \frac{x}{L} \right\rfloor \times \{d\}$$

1.3 Two-nodes bending bar element (Beam element in the x-y plane)
Elément barre de flexion à deux nœuds (Elément poutre dans le plan x-y)

1.3.1 Standard element: (Elément standard)

$$U = \begin{Bmatrix} v \\ \theta \end{Bmatrix} \qquad et \ \{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \ with \ \theta = \theta_z = \frac{dv}{dx}$$

$$V = [N] \times \{d\} = \left\lfloor 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad x - \frac{2x^2}{L} + \frac{x^3}{L^2} \quad \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad - \frac{x^2}{L} + \frac{x^3}{L^2} \right\rfloor \times \{d\}$$

$$\theta = \frac{dV}{dx} = [N'] \times \{d\} = \left\lfloor -\frac{6x}{L^2} + \frac{6x^2}{L^3} \quad 1 - \frac{4x}{L} + \frac{3x^2}{L^2} \quad \frac{6x}{L^2} - \frac{6x^2}{L^3} \quad -\frac{2x}{L} + \frac{3x^2}{L^2} \right\rfloor \times \{d\}$$

1.3.2 Left hinged beam element : (Elément articulée à gauche)

$$v = [N] \times \{d\} = \left\lfloor 1 - \frac{3x}{2L} + \frac{x^3}{2L^3} - \quad 0 \quad \frac{3x}{2L} - \frac{x^3}{2L^3} \quad -\frac{x}{2} + \frac{x^3}{2L^2} \right\rfloor \times \{d\}$$

$$\theta = \frac{dv}{dx} = [N'] \times \{d\} = \left\lfloor -\frac{3}{2L} + \frac{3x^2}{2L^3} \quad 0 \quad \frac{3}{2L} - \frac{3x^2}{2L^3} \quad -\frac{1}{2} + \frac{3x^2}{2L^2} \right\rfloor \times \{d\}$$

*1.3.3 **Right hinged beam element:*** *(Elément articulé à droite)*

$$\boldsymbol{v} = [N] \times \{\boldsymbol{d}\} = \left\lfloor \mathbf{1} - \frac{3x^2}{2L^2} + \frac{x^3}{2L^3} \quad x - \frac{3x^2}{2L} + \frac{x^3}{2L^2} \quad \frac{3x^2}{2L^2} - \frac{x^3}{2L^3} \quad 0 \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{\theta} = \frac{dv}{dx} = [N'] \times \{\boldsymbol{d}\} = \left\lfloor -\frac{3x}{L^2} + \frac{3x^2}{2L^3} \quad \mathbf{1} - \frac{3x}{L} + \frac{3x^2}{2L^2} \quad \frac{3x}{L^2} - \frac{3x^2}{2L^3} \quad \mathbf{0} \right\rfloor \times \{\boldsymbol{d}\}$$

*1.3.4 **Hinged beam element at its two nodes:*** *(Elément articulé à ses deux nœuds)*

$$\boldsymbol{v} = [N] \times \{\boldsymbol{d}\} = \left\lfloor \mathbf{1} - \frac{x}{L} \quad \mathbf{0} \quad \frac{x}{L} \quad 0 \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{\theta} = \frac{dv}{dx} = [N'] \times \{\boldsymbol{d}\} = \left\lfloor -\frac{1}{L} \quad \mathbf{0} \quad \frac{1}{L} \quad \mathbf{0} \right\rfloor \times \{\boldsymbol{d}\}$$

1.4 Two-nodes tension–compression and bending bar element (2D frame element in the x-y plane): Elément barre de traction –compression et de flexion à deux nœuds (Elément portique 2D dans le plan x-y)

$$\boldsymbol{U} = \begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} \; et \, \{\boldsymbol{d}\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}, \quad \boldsymbol{\theta} = \boldsymbol{\theta_z}$$

1.4.1 Standard 2D Frame element:

$$\boldsymbol{u} = \left\lfloor \mathbf{1} - \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \quad \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{w} = \left\lfloor \mathbf{0} \quad \mathbf{1} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad x - \frac{2x^2}{L} + \frac{x^3}{L^2} \quad \mathbf{0} \quad \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad -\frac{x^2}{L} + \frac{x^3}{L^2} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{\theta} = -\frac{dw}{dx} = \left\lfloor \mathbf{0} \quad -\frac{6x}{L^2} + \frac{6x^2}{L^3} \quad \mathbf{1} - \frac{4x}{L} + \frac{3x^2}{L^2} \quad \mathbf{0} \quad \frac{6x}{L^2} - \frac{6x^2}{L^3} \quad -\frac{2x}{L} + \frac{3x^2}{L^2} \right\rfloor \times \{\boldsymbol{d}\}$$

1.4.2 Left hinged 2D Frame element:

$$\boldsymbol{u} = \left\lfloor \mathbf{1} - \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \quad \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{v} = [N] \times \{\boldsymbol{d}\} = \left\lfloor 0 \quad \mathbf{1} - \frac{3x}{2L} + \frac{x^3}{2L^3} \quad 0 \quad 0 \quad \frac{3x}{2L} - \frac{x^3}{2L^3} \quad -\frac{x}{2} + \frac{x^3}{2L^2} \right\rfloor \times \{\boldsymbol{d}\}$$

$$\boldsymbol{\theta} = \frac{dv}{dx} = [N'] \times \{\boldsymbol{d}\} = \left\lfloor \mathbf{0} \quad -\frac{3}{2L} + \frac{3x^2}{2L^3} \quad \mathbf{0} \quad \mathbf{0} \quad \frac{3}{2L} - \frac{3x^2}{2L^3} \quad -\frac{1}{2} + \frac{3x^2}{2L^2} \right\rfloor \times \{\boldsymbol{d}\}$$

1.4.3 Right hinged 2D Frame element:

$$\boldsymbol{u} = \left\lfloor \mathbf{1} - \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \quad \tfrac{x}{L} \quad \mathbf{0} \quad \mathbf{0} \right\rfloor \times \{\boldsymbol{d}\}$$

$$v = [N] \times \{d\} = \left\lfloor 0 \quad 1 - \frac{3x^2}{2L^2} + \frac{x^3}{2L^3} \quad x - \frac{3x^2}{2L} + \frac{x^3}{2L^2} \quad 0 \quad \frac{3x^2}{2L^2} - \frac{x^3}{2L^3} \quad 0 \right\rfloor \times \{d\}$$

$$\theta = \frac{dv}{dx} = [N'] \times \{d\} = \left\lfloor 0 \quad -\frac{3x}{L^2} + \frac{3x^2}{2L^3} \quad 1 - \frac{3x}{L} + \frac{3x^2}{2L^2} \quad 0 \quad \frac{3x}{L^2} - \frac{3x^2}{2L^3} \quad 0 \right\rfloor \times \{d\}$$

1.4.4 Hinged 2D Frame element at its two nodes:

$$u = \left\lfloor 1 - \frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \quad 0 \right\rfloor \times \{d\}$$

$$v = [N] \times \{d\} = \left\lfloor 0 \quad 1 - \frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \right\rfloor \times \{d\}$$

$$\theta = \frac{dv}{dx} = [N'] \times \{d\} = \left\lfloor 0 \quad -\frac{1}{L} \quad 0 \quad 0 \quad \frac{1}{L} \quad 0 \right\rfloor \times \{d\}$$

1.5. Bending and Torsion Bar Element (crossed beam element)

$$U = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix}, \qquad \{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \end{Bmatrix},$$

Standard *Crossed Beam* element:

$$w = \left\lfloor 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad 0 \quad -x + \frac{2x^2}{L} - \frac{x^3}{L^2} \quad \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad 0 \quad \frac{x^2}{L} - \frac{x^3}{L^2} \right\rfloor \times \{d\}$$

$$\theta_x = \left\lfloor 0 \quad 1 - \frac{x}{L} \quad 0 \quad 0 \quad \frac{x}{L} \quad 0 \right\rfloor \times \{d\}$$

$$\theta_y = -\frac{dw}{dx} = \left\lfloor \frac{6x}{L^2} - \frac{6x^2}{L^3} \quad 0 \quad 1 - \frac{4x}{L} + \frac{3x^2}{L^2} \quad \frac{6x}{L^2} - \frac{6x^2}{L^3} \quad 0 \quad -\frac{2x}{L} + \frac{3x^2}{L^2} \right\rfloor \times \{d\}$$

1.6. Three nodes Plane Elasticity Triangular element *(Élément triangulaire d'élasticité plane à trois nœuds)*

$$U = \begin{Bmatrix} u \\ v \end{Bmatrix}, \{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}$$

$$\boldsymbol{N_1} = \frac{\mathbf{1}}{\mathbf{2\Delta}}[(\boldsymbol{x_2 y_3} - \boldsymbol{y_2 x_3}) + \boldsymbol{x}(\boldsymbol{y_2} - \boldsymbol{y_3}) + \boldsymbol{y}(\boldsymbol{x_3} - \boldsymbol{x_2})]$$

$$\boldsymbol{N_2} = \frac{\mathbf{1}}{\mathbf{2\Delta}}[(\boldsymbol{y_1 x_3} - \boldsymbol{x_1 y_3}) + \boldsymbol{x}(\boldsymbol{y_3} - \boldsymbol{y_1}) + \boldsymbol{y}(\boldsymbol{x_1} - \boldsymbol{x_3})]$$

$$\boldsymbol{N_3} = \frac{\mathbf{1}}{\mathbf{2\Delta}}[(\boldsymbol{x_1 y_2} - \boldsymbol{y_1 x_2}) + \boldsymbol{x}(\boldsymbol{y_1} - \boldsymbol{y_2}) + \boldsymbol{y}(\boldsymbol{x_2} - \boldsymbol{x_1})]$$

Δ *is the area of the triangular element and* $\{x_i, y_i\}$ *are the coordinates of its nodes*

$$2\Delta = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

1.7 Tetrahedral element (Volumic four nodes element):

$$U = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, \{d\} = \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6 \quad d_7 \quad d_8 \quad d_9 \quad d_{10} \quad d_{11} \quad d_{12} \rfloor^T$$

$$\{d\} = \lfloor u_1 \quad v_1 \quad w_1 \quad u_2 \quad v_2 \quad w_2 \quad u_3 \quad v_3 \quad w_3 \quad u_4 \quad v_4 \quad w_4 \rfloor^T$$

$$U = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \{d\}$$

$$\boldsymbol{N_i} = \boldsymbol{L_i} = \frac{\mathbf{1}}{\mathbf{6V}}(\boldsymbol{a_i} + \boldsymbol{b_i x} + \boldsymbol{c_i y} + \boldsymbol{d_i z})$$

Where:

V *is the volume of the tetrahedral element*

$$6V = det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

$$a_i = c_{1i}{}^T \; ; \; b_i = c_{2i}{}^T \quad ; \quad c_i = c_{3i}{}^T \; ; \; d_i = c_{4i}{}^T$$

$$c_{11} = det\begin{bmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix};\; c_{12} = det\begin{bmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix};\; c_{13} = det\begin{bmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{14} = det\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

2 Stiffness matrices of some elements: (Matrices de rigidités de quelques éléments)

Let [T], [k], and [K] denote respectively the transformation matrix, the stiffness matrix in the local frame, and the stiffness matrix in the global frame. ((Notons respectivement par $[T]$, $[k]$ et $[K]$ la matrice de transformation, la matrice de rigidité dans le repère local et la matrice de rigidité dans le repère global)

2.1 Spring element : Elément ressort :

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{l} & \boldsymbol{m} \end{bmatrix} \qquad [\boldsymbol{k}] = \boldsymbol{k}\begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix} \qquad [\boldsymbol{K}] = \boldsymbol{k}\begin{bmatrix} \boldsymbol{l^2} & \boldsymbol{lm} & \boldsymbol{-l^2} & \boldsymbol{-lm} \\ \boldsymbol{lm} & \boldsymbol{m^2} & \boldsymbol{-lm} & \boldsymbol{-m^2} \\ \boldsymbol{-l^2} & \boldsymbol{-lm} & \boldsymbol{l^2} & \boldsymbol{lm} \\ \boldsymbol{-lm} & \boldsymbol{-m^2} & \boldsymbol{lm} & \boldsymbol{m^2} \end{bmatrix}$$

2.2 Two-node tension–compression bar element:

(Elément barre de traction compression à deux nœuds)

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{l} & \boldsymbol{m} \end{bmatrix} \qquad [\boldsymbol{k}] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{-1} & \boldsymbol{1} \end{bmatrix} \qquad K] = \frac{\boldsymbol{EA}}{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{l^2} & \boldsymbol{lm} & \boldsymbol{-l^2} & \boldsymbol{-lm} \\ \boldsymbol{lm} & \boldsymbol{m^2} & \boldsymbol{-lm} & \boldsymbol{-m^2} \\ \boldsymbol{-l^2} & \boldsymbol{-lm} & \boldsymbol{l^2} & \boldsymbol{lm} \\ \boldsymbol{-lm} & \boldsymbol{-m^2} & \boldsymbol{lm} & \boldsymbol{m^2} \end{bmatrix}$$

*2.3. Bending bar element (Beam element) (*Elément barre de flexion (Elément poutre))

in the x-y plane: $\boldsymbol{I = I_z}, \boldsymbol{\theta = \theta_z}$

2.3.1 Standard Beam element:

$$[\boldsymbol{k}] = [\boldsymbol{K}] = \frac{\boldsymbol{EI}}{\boldsymbol{L^3}}\begin{bmatrix} \boldsymbol{12} & \boldsymbol{6L} & \boldsymbol{-12} & \boldsymbol{6L} \\ \boldsymbol{6L} & \boldsymbol{4L^2} & \boldsymbol{-6L} & \boldsymbol{2L^2} \\ \boldsymbol{-12} & \boldsymbol{-6L} & \boldsymbol{12} & \boldsymbol{-6L} \\ \boldsymbol{6L} & \boldsymbol{2L^2} & \boldsymbol{-6L} & \boldsymbol{4L^2} \end{bmatrix}$$

2.3.2 Left hinged Beam element:

$$\theta_1 = -\frac{3}{2L}v_1 + \frac{3}{2L}v_2 - \frac{1}{2}\theta_2$$

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & 0 & -1 & L \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -L \\ L & 0 & -L & L^2 \end{bmatrix}$$

2.3.3 Right hinged Beam element:

$$\theta_2 = -\frac{3}{2L}v_1 + \frac{3}{2L}v_2 - \frac{1}{2}\theta_1$$

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & L & -1 & 0 \\ L & L^2 & -L & 0 \\ -1 & -L & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.3.4 Beam element hinged at its two nodes:

$$\theta_1 = \frac{-v_1}{L} + \frac{v_2}{L} = \theta_2$$

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.4 Bending and tension compression barre element (2D frame element) *(Elément barre se flexion et traction compression (Elément portique 2D))*

In the x-y local coordinates: $\boldsymbol{I = I_z}$, $\boldsymbol{\theta = \theta_z}$

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{-m} & \boldsymbol{l} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{l} & \boldsymbol{m} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-m} & \boldsymbol{l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

2.4.1 Standard 2D Frame element:

$$[\boldsymbol{k}]=\frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I}{L^2} & \frac{6I}{L} & 0 & \frac{-12I}{L^2} & \frac{6I}{L} \\ 0 & \frac{6I}{L} & 4I & 0 & \frac{-6I}{L} & 2I \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{-12I}{L^2} & \frac{-6I}{L} & 0 & \frac{12I}{L^2} & \frac{-6I}{L} \\ 0 & \frac{6I}{L} & 2I & 0 & \frac{-6I}{L} & 4I \end{bmatrix}$$

$$[\mathrm{K}]=\frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} +\left(\mathrm{A}l^2+\frac{12\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & -\frac{6\mathrm{mI}}{\mathrm{L}} & -\left(\mathrm{A}l^2+\frac{12\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & -\frac{6\mathrm{mI}}{\mathrm{L}} \\ +l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{12\mathrm{I}}{\mathrm{L}^2}l^2\right) & \frac{6l\mathrm{I}}{\mathrm{L}} & -l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{12\mathrm{I}}{\mathrm{L}^2}l^2\right) & \frac{6l\mathrm{I}}{\mathrm{L}} \\ -\frac{6\mathrm{mI}}{\mathrm{L}} & \frac{6l\mathrm{I}}{\mathrm{L}} & 4\mathrm{I} & \frac{6\mathrm{mI}}{\mathrm{L}} & -\frac{6l\mathrm{I}}{\mathrm{L}} & 2\mathrm{I} \\ -\left(\mathrm{A}l^2+\frac{12\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & \frac{6\mathrm{mI}}{\mathrm{L}} & +\left(\mathrm{A}l^2+\frac{12\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & \frac{6\mathrm{mI}}{\mathrm{L}} \\ -l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{12\mathrm{I}}{\mathrm{L}^2}l^2\right) & -\frac{6l\mathrm{I}}{\mathrm{L}} & +l\mathrm{m}\left(\mathrm{A}-\frac{12\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{12\mathrm{I}}{\mathrm{L}^2}l^2\right) & -\frac{6l\mathrm{I}}{\mathrm{L}} \\ -\frac{6\mathrm{mI}}{\mathrm{L}} & \frac{6l\mathrm{I}}{\mathrm{L}} & 2\mathrm{I} & \frac{6\mathrm{mI}}{\mathrm{L}} & -\frac{6l\mathrm{I}}{\mathrm{L}} & 4\mathrm{I} \end{bmatrix}$$

2.4.2 Left hinged 2D Frame element

$$[\boldsymbol{k}]=\frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{3I}{L^2} & 0 & 0 & \frac{3I}{L^2} & \frac{3I}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{-3I}{L^2} & 0 & 0 & \frac{3I}{L^2} & \frac{-3I}{L} \\ 0 & \frac{3I}{L} & 0 & 0 & \frac{-3I}{L} & 3I \end{bmatrix}$$

$$[\mathrm{K}]=\frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0 & -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\frac{3\mathrm{mI}}{\mathrm{L}} \\ +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0 & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & \frac{3l\mathrm{I}}{\mathrm{L}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0 & +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & \frac{3\mathrm{mI}}{\mathrm{L}} \\ -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0 & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & -\frac{3l\mathrm{I}}{\mathrm{L}} \\ -\frac{3\mathrm{mI}}{\mathrm{L}} & \frac{3l\mathrm{I}}{\mathrm{L}} & 0 & \frac{3\mathrm{mI}}{\mathrm{L}} & -\frac{3l\mathrm{I}}{\mathrm{L}} & 3\mathrm{I} \end{bmatrix}$$

2.4.3 Right hinged 2D Frame element

$$[\boldsymbol{k}]=\frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0\\ 0 & \frac{3I}{L^2} & \frac{3I}{L} & 0 & \frac{-3I}{L^2} & 0\\ 0 & \frac{3I}{L} & 3I & 0 & \frac{-3I}{L} & 0\\ -A & 0 & 0 & A & 0 & 0\\ 0 & \frac{-3I}{L^2} & \frac{-3I}{L} & 0 & \frac{3I}{L^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]=\frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\frac{3\mathrm{mI}}{\mathrm{L}} & -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0\\ +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & \frac{3l\mathrm{I}}{\mathrm{L}} & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0\\ -\frac{3\mathrm{mI}}{\mathrm{L}} & \frac{3l\mathrm{I}}{\mathrm{L}} & 3\mathrm{I} & \frac{3\mathrm{mI}}{\mathrm{L}} & -\frac{3l\mathrm{I}}{\mathrm{L}} & 0\\ -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & \frac{3\mathrm{mI}}{\mathrm{L}} & +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0\\ -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & -\frac{3l\mathrm{I}}{\mathrm{L}} & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.4.5 2D Frame element hinged at its two nodes

$$[\boldsymbol{k}]=\frac{E}{L}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0\\ 0 & \frac{3I}{L^2} & 0 & 0 & \frac{-3I}{L^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -A & 0 & 0 & A & 0 & 0\\ 0 & \frac{-3I}{L^2} & 0 & 0 & \frac{3I}{L^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{K}]=\frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0 & -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0\\ +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0 & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0 & +\left(\mathrm{A}l^2+\frac{3\mathrm{I}}{\mathrm{L}^2}\mathrm{m}^2\right) & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & 0\\ -l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & -\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0 & +l\mathrm{m}\left(\mathrm{A}-\frac{3\mathrm{I}}{\mathrm{L}^2}\right) & +\left(\mathrm{Am}^2+\frac{3\mathrm{I}}{\mathrm{L}^2}l^2\right) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.5. Bending and Torsion Bar Element (crossed beam element): *(Elément barre de torsion et de flexion (Elément poutre croisée)*

Standard Crossed Beam element

$$[T]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l & m & 0 & 0 & 0 \\ 0 & -m & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & l & m \\ 0 & 0 & 0 & 0 & -m & l \end{bmatrix}$$

$$[\boldsymbol{k}]=\frac{\boldsymbol{E}}{\boldsymbol{L}}\begin{bmatrix} \frac{12I_y}{L^2} & 0 & \frac{-6I_y}{L} & \frac{-12I_y}{L^2} & 0 & \frac{-6I_y}{L} \\ 0 & \frac{J}{2(1+v)} & 0 & 0 & \frac{-J}{2(1+v)} & 0 \\ \frac{-6I_y}{L} & 0 & 4I_y & \frac{6I_y}{L} & 0 & 2I_y \\ \frac{-12I}{L^2} & 0 & \frac{6I}{L} & \frac{12I_y}{L^2} & 0 & \frac{6I_y}{L} \\ 0 & \frac{-J}{2(1+v)} & 0 & 0 & \frac{J}{2(1+v)} & 0 \\ \frac{-6I}{L} & 0 & 2I_y & \frac{6I_y}{L} & 0 & 4I_y \end{bmatrix}$$

$$[\boldsymbol{K}]=\frac{\boldsymbol{E}}{\boldsymbol{L}}\begin{bmatrix} \frac{12I}{L^2} & \frac{6mI}{L} & \frac{-6lI}{L} & \frac{-12I}{L^2} & \frac{6mI}{L} & \frac{-6lI}{L} \\ \frac{6mI}{L} & \left(\frac{l^2J}{2(1+v)}+4m^2I\right) & lm\left(\frac{J}{2(1+v)}-4I\right) & \frac{-6mI}{L} & \left(\frac{-l^2J}{2(1+v)}+2m^2I\right) & -lm\left(\frac{J}{2(1+v)}+2I\right) \\ \frac{-6lI}{L} & lm\left(\frac{J}{2(1+v)}-4I\right) & \left(\frac{m^2J}{2(1+v)}+4l^2I\right) & \frac{6lI}{L} & -lm\left(\frac{J}{2(1+v)}+2I\right) & \left(\frac{-m^2J}{2(1+v)}+2l^2I\right) \\ \frac{-12I}{L^2} & \frac{-6mI}{L} & \frac{6lI}{L} & \frac{12I}{L^2} & \frac{-6mI}{L} & \frac{6lI}{L} \\ \frac{6mI}{L} & \left(\frac{-l^2J}{2(1+v)}+2m^2I\right) & -lm\left(\frac{J}{2(1+v)}+2I\right) & -\frac{6mI}{L} & \left(\frac{l^2J}{2(1+v)}+4m^2I\right) & lm\left(\frac{J}{2(1+v)}-4I\right) \\ \frac{-6lI}{L} & -lm\left(\frac{J}{2(1+v)}+2I\right) & \left(\frac{-m^2J}{2(1+v)}+2l^2I\right) & \frac{6lI}{L} & lm\left(\frac{J}{2(1+v)}-4I\right) & \left(\frac{m^2J}{2(1+v)}+4l^2I\right) \end{bmatrix}$$

1.6. Three nodes Plane Elasticity Triangular element *(Élément triangulaire d'élasticité plane à trois nœuds)*

$$[\boldsymbol{k}]=[\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}]\Delta\boldsymbol{t}$$

Where: Δ *is the area of the triangular element and* ***t*** *is its thickness*

$$[\boldsymbol{B}] = \frac{\boldsymbol{1}}{\boldsymbol{2\Delta}}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

$$[\boldsymbol{C}] = \frac{\boldsymbol{E}}{\boldsymbol{1 - v^2}}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1 - v}{2} \end{bmatrix} \textit{ in the plane stresses case}$$

$$[\boldsymbol{C}] = \frac{\boldsymbol{E}}{\boldsymbol{(1 + v)(1 - 2v)}}\begin{bmatrix} 1 - v & v & 0 \\ v & 1 - v & 0 \\ 0 & 0 & \frac{1 - 2v}{2} \end{bmatrix} \textit{ in the plane deformations case}$$

1.7 Tetrahedral element (Volumic four nodes element): *(Elément tétraédrique à 4 nœuds)*

$$[\boldsymbol{k}] = \boldsymbol{V}[\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}]$$

$$[B] = \begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ 0 & d_1 & c_1 & 0 & d_2 & c_2 & 0 & d_3 & c_3 & 0 & d_4 & c_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \end{bmatrix}$$

$$[C] = \frac{\boldsymbol{E}}{\boldsymbol{(1 - 2\nu)(1 + \nu)}}\begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1 - 2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1 - 2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}$$