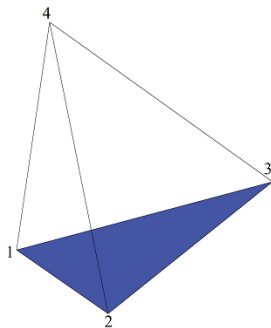


Chapter 4 : Finite element in three dimensions

4.1 Introduction

Three-dimensional finite elements (solid elements) are used to solve problems described by partial differential equations with three variables. In solid mechanics problems, there are a total of six possible stress components: three normal stresses and three shear stresses, which must be considered. Generally, a 3D solid element can be tetrahedral or hexahedral in shape, with flat or curved surfaces. Each node of the element will have three translational degrees of freedom (3DOF). Thus, the element can deform in all three spatial directions

4.2 Tetrahedral element (4 Nodes; 4 faces):



4.2.1. Displacements field:

$$\{d\} = [u_1 \quad v_1 \quad w_1 \quad u_2 \quad v_2 \quad w_2 \quad u_3 \quad v_3 \quad w_3 \quad u_4 \quad v_4 \quad w_4]^T$$

$$u = N_1 U_1 + N_2 U_2 + N_3 U_3 + N_4 U_4 ; \quad \text{likewise } v \text{ and } w$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \{d\}$$

The shape functions N_i can be determined by approximating each component with a 4-term polynomial:

$$u = a_0 + a_1x + a_2y + a_3z = [1 \quad x \quad y \quad z]\{a\}$$

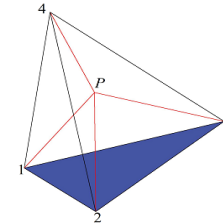
and using the boundary conditions that lead to the system:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \{a\}$$

Solving this system using Cramer's method gives the a_i in terms of the u_i , and by substituting into the expression for u ; we can determine the N_i .

However, we prefer to use the volume coordinates L :

$$L_1 = \frac{V_{p234}}{V}, \quad L_2 = \frac{V_{p134}}{V}, \quad L_3 = \frac{V_{p124}}{V}, \quad L_4 = \frac{V_{p123}}{V}$$



Where V is the volume of the element and V_{pijk} is the volume of the tetrahedron whose vertices are an interior point p and the vertices i, j, k of the element.

We deduce that:

$$L_1 + L_2 + L_3 + L_4 = 1; \quad V_{p234} + V_{p134} + V_{p124} + V_{p123} = V$$

If p becomes one of the vertices i of the tetrahedron we will have $L_i = 1$ which allows one to write:

$$x = L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4$$

$$y = L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4$$

$$z = L_1z_1 + L_2z_2 + L_3z_3 + L_4z_4$$

From which we can form the following system:

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix}$$

Let's use the cofactor method to determine the inverse matrix:

For a matrix $[A]$: $[A]^{-1} = \frac{1}{\det[A]} [\text{com}[A]]^T$

The terms of: $\text{com} \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$ are:

$$c_{11} = \det \begin{bmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix}; \quad c_{12} = \det \begin{bmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix}; \quad c_{13} = \det \begin{bmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{14} = \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{21} = \det \begin{bmatrix} 1 & 1 & 1 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{bmatrix}; c_{22} = \det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{bmatrix}; c_{23} = \det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{24} = \det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{31} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \end{bmatrix}; c_{32} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ z_1 & z_3 & z_4 \end{bmatrix}; c_{33} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ z_1 & z_2 & z_4 \end{bmatrix}$$

$$c_{34} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$c_{41} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{bmatrix}; c_{42} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{bmatrix}; c_{43} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{bmatrix}$$

$$c_{44} = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Noting that:

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} = 6V$$

Then:

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

with:

$$a_i = c_{1i}^T; b_i = c_{2i}^T; c_i = c_{3i}^T; d_i = c_{4i}^T$$

$$N_i = L_i = \frac{1}{6V} (a_i + b_i x + c_i y + d_i z)$$

4.2.2 Deformations field:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{Bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\varepsilon = [B]\{d\} = \begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \end{bmatrix} \{d\}$$

4.2.3. Stresses field:

$$\sigma = [C]\varepsilon$$

$$\sigma = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

4.2.4. Stiffness matrix of the element:

$$[k] = \int_V [B]^T [C] [B] dV$$

As $[B]$ and $[C]$ are constants, then:

$$[k] = V[B]^T [C] [B]$$