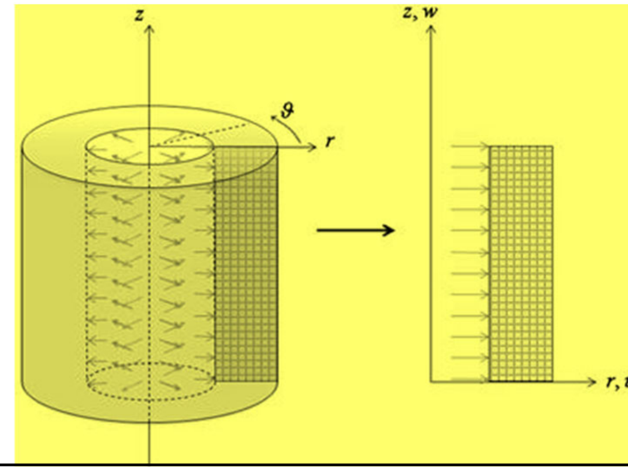
3.4. Two-dimensional element for the study of axisymmetric problems:

The symmetry is considered around the z-axis:

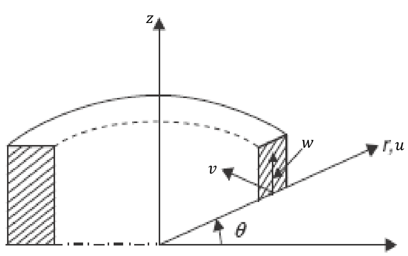
Due to this symmetry, every vertical section in the cylinder undergoes the same stresses and deformations, independently of θ; therefore, the problem is reduced to a plane problem in the r - z plane



Let's use the cylindrical coordinates r, θ, z related to the coordinates x, y and z by :

$$x = r\cos\theta,\ \ y = r\sin\theta\ ,\quad z = z$$

And denote the displacements in this reference system as: $u\ , v\ et\ \ w$



For the triangular element e will have:

3.4.1. Displacement field: $v\ =\ 0$

$$\begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} where \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

With the same shape functions

$$N_1 = \frac{1}{2\Delta}[(r_2z_3 - z_2r_3) + r(z_2 - z_3) + z(r_3 - r_2\ )]$$

$$N_2 = \frac{1}{2\Delta}[(z_1r_3 - r_1z_3) + r(z_3 - z_1) + z(r_1 - r_3\ )]$$

$$N_3 = \frac{1}{2\Delta}[(\ r_1z_2 - z_1r_2) + r(z_1 - z_2) + z(r_2 - r_1\ )]$$

3.4.2. Deformation field:

The deformations are:

$$\varepsilon_r = \frac{\partial u}{\partial r};\ \varepsilon_\theta = \frac{u}{r};\ \varepsilon_z = \frac{\partial w}{\partial z};\ \gamma_{r\theta} = \gamma_{\theta z} = 0;\ \gamma_{rz} =;\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} = [B]\{d\}$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} z_2 - z_3 & 0 & z_3 - z_1 & 0 & z_1 - z_2 & 0 \\ 0 & r_3 - r_2 & 0 & r_1 - r_3 & 0 & r_2 - r_1 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ r_3 - r_2 & z_2 - z_3 & r_1 - r & z_3 - z_{13} & r_2 - r_1 & z_1 - z_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

3.4.3 Stresses feild:

$$\boldsymbol{\sigma} = [\boldsymbol{C}]\boldsymbol{\varepsilon}$$

$$[\boldsymbol{C}] = \frac{\boldsymbol{E}}{(\mathbf{1}+\boldsymbol{v})(\mathbf{1}-\mathbf{2}\boldsymbol{v})} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

3.4.4 Stiffness matrix :

$$[k] = \int_V [\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}]\,[\boldsymbol{B}].\boldsymbol{dv} = \int_0^{2\pi} \boldsymbol{d\theta} \iint_{rz} [\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}].\boldsymbol{dr}.\boldsymbol{dz}$$

$$[k] = 2\pi \iint_{rz} [\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}].\boldsymbol{r}.\boldsymbol{dr}.\boldsymbol{dz}$$

There are two possibilities for evaluating this matrix:

The first is the explicit evaluation of the product $[\boldsymbol{B}]^{\boldsymbol{T}}[\boldsymbol{C}][\boldsymbol{B}]$ *and then the term-by-term integration;*

The second is numerical integration. The simplest numerical integration procedure consists of evaluating all quantities for a centroid

$$\bar{r} = \frac{r_i + r_j + r_m}{3} \quad ; \quad \bar{z} = \frac{z_i + z_j + z_m}{3}$$

In this case::

$$[\boldsymbol{k}] = \mathbf{2}\boldsymbol{\pi}[\overline{\boldsymbol{B}}]^{\boldsymbol{T}}[\boldsymbol{C}][\overline{\boldsymbol{B}}]\bar{r}\Delta$$

Where : $[\overline{\boldsymbol{B}}]$ *is the evaluation of* $[\boldsymbol{B}]$ *at the element centroid point..*