

### ***Chapter 3 : Finite element in two dimensions***

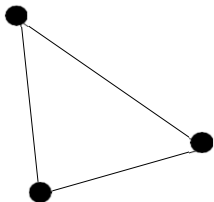
*Two-dimensional finite elements are used to solve problems described by partial differential equations with two variables. Many practical engineering problems can be conceptually defined in two dimensions and can be described by two-dimensional partial differential equations, especially when the primary unknowns vary mainly in two directions while remaining constant in the third direction.*

*Plane elasticity, axisymmetric problems, and two-dimensional fluid flow are examples of problems that can be studied using two-dimensional finite elements."*

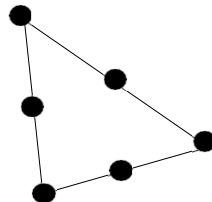
#### ***3.1. Commonly used elements and shape functions:***

*Triangular element with 3, 6, or 9 nodes*

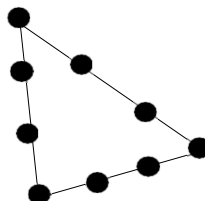
*Linear element  
with 3 nodes*



*Quadratic element  
with 6 nodes*

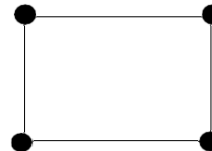


*Cubic element  
with 9 nodes*

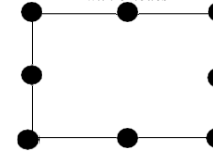


***Quadrilateral elements with 4, 8, or 12 nodes (especially the rectangular element)***

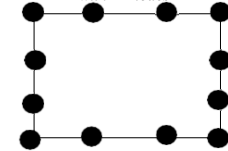
*Linear element  
with 4 nodes*



*Quadratic element  
with 8 nodes*

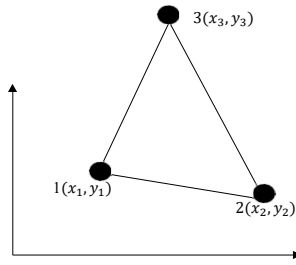


*Cubic element  
with 12 nodes*



### 3.2. Shape functions:

#### 3.2.1. Triangular element:



Each component will be approximated separately:

$$u(x, y) = a_0 + a_1x + a_2y = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}$$

Boundary conditions are:

$$\text{Node 1 : } (x, y) = (x_1, y_1) \rightarrow u(x, y) = u_1 = a_0 + a_1x_1 + a_2y_1$$

$$\text{Node 2 : } (x, y) = (x_2, y_2) \rightarrow u(x, y) = u_2 = a_0 + a_1x_2 + a_2y_2$$

$$\text{Node 3 : } (x, y) = (x_3, y_2) \rightarrow u(x, y) = u_3 = a_0 + a_1x_3 + a_2y_3$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}, \text{ In condensed form } \{d\} = [\bar{P}]\{a\}$$

To determine  $\{a\}$  we invert  $[\bar{P}]$ , but we prefer to solve the system using Cramer's method.

It should be noted that the determinant of  $[\bar{P}]$  is not zero since it is equal to twice the area of triangle 1 2 3. Let's denote its value by  $2\Delta$ , Thus, the solution will be

$$a_0 = \frac{\begin{vmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{1}{2\Delta} ((x_2y_3 - y_2x_3)u_1 - x_1(u_2y_3 - y_2u_3) + y_1(u_2x_3 - x_2u_3))$$

$$a_0 = \frac{1}{2\Delta} ((x_2y_3 - y_2x_3)u_1 + (y_1x_3 - x_1y_3)u_2 + (x_1y_2 - y_1x_2)u_3)$$

$$a_1 = \frac{\begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{1}{2\Delta} ((y_3u_2 - u_3y_2) - u_1(y_3 - y_2) + y_1(u_3 - u_2))$$

$$a_1 = \frac{1}{2\Delta} ((y_2 - y_3)u_1 + (y_3 - y_1)u_2 + (y_1 - y_2)u_3)$$

$$a_2 = \frac{\begin{vmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}} = \frac{1}{2\Delta} ((x_2u_3 - u_2x_3) - x_1(u_3 - u_2) + u_1(x_3 - x_2))$$

$$a_2 = \frac{1}{2\Delta} ((x_3 - x_2)u_1 + (x_1 - x_3)u_2 + (x_2 - x_1)u_3)$$

Hence :

$$\{a\} = [\bar{P}]^{-1}\{u\} \rightarrow \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} x_2y_3 - y_2x_3 & y_1x_3 - x_1y_3 & x_1y_2 - y_1x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [P]\{a\} = \begin{bmatrix} 1 & x & y \end{bmatrix} \times \frac{1}{2\Delta} \begin{bmatrix} x_2y_3 - y_2x_3 & y_1x_3 - x_1y_3 & x_1y_2 - y_1x_2 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = N_1u_1 + N_2u_2 + N_3u_3$$

Where :

$$N_1 = \frac{1}{2\Delta} [(x_2y_3 - y_2x_3) + x(y_2 - y_3) + y(x_3 - x_2)]$$

$$N_2 = \frac{1}{2\Delta} [(y_1x_3 - x_1y_3) + x(y_3 - y_1) + y(x_1 - x_3)]$$

$$N_3 = \frac{1}{2\Delta} [(x_1y_2 - y_1x_2) + x(y_1 - y_2) + y(x_2 - x_1)]$$

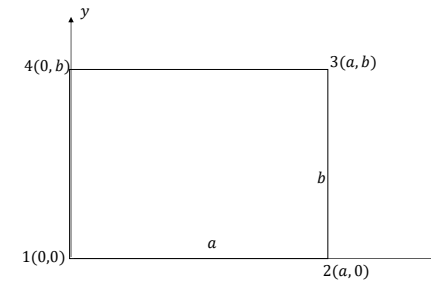
A similar reasoning for the component v gives :

$$v = N_1v_1 + N_2v_2 + N_3v_3$$

The complete field is written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix}, \quad \text{where} \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

### 3.2.2. Rectangular element



$$u(x, y) = a_0 + a_1x + a_2y + a_3xy = [1 \quad x \quad y \quad xy] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Boundary conditions are:

Node 1 :  $(x, y) = (0, 0) \rightarrow u(x, y) = u_1 = a_0$

Node 2 :  $(x, y) = (a, 0) \rightarrow u(x, y) = u_2 = a_0 + a_1a$

Node 3 :  $(x, y) = (a, b) \rightarrow u(x, y) = u_3 = a_0 + a_1a + a_2b + a_3ab$

Node 4 :  $(x, y) = (0, b) \rightarrow u(x, y) = u_4 = a_0 + a_2b$

$$a_0 = u_1, \quad a_1 = \frac{1}{a}(u_2 - u_1), \quad a_2 = \frac{1}{b}(u_4 - u_1), \quad a_3 = \frac{1}{ab}(u_3 + u_1 - u_2 - u_4)$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 & 0 \\ -b & b & 0 & 0 \\ -a & 0 & 0 & a \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u(x, y) = [1 \quad x \quad y \quad xy] \times \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 & 0 \\ -b & b & 0 & 0 \\ -a & 0 & 0 & a \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

Where :

$$N_1 = \frac{1}{ab}(ab - bx - ay + xy) = 1 - \frac{x}{a} - \frac{y}{b} + \frac{xy}{ab}$$

$$N_2 = \frac{1}{ab}(bx - xy) = \frac{x}{a} - \frac{xy}{ab}$$

$$N_3 = \frac{xy}{ab}$$

$$N_4 = \frac{1}{ab}(ay - xy) = \frac{y}{b} - \frac{xy}{ab}$$

A similar reasoning for the component v gives:

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

The full field is written as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix}, \quad \text{avec} \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_4 \end{Bmatrix}$$