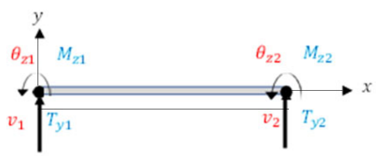
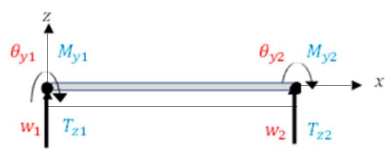
2.3 Bending bar element (Beam element)

continuous beam structures

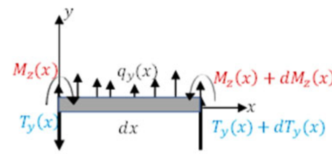
2.3.1. Definition, notations, and sign conventions:



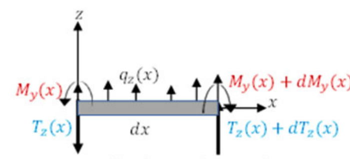


2.3.2. Basic Equations

(see strength materials, Elasticity and structural mechanic)



Bending in the x-y plane



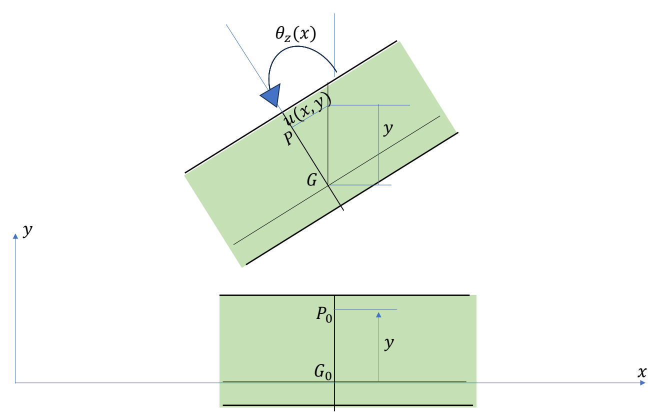
Bending in the x-z plane

For the bending in the x-y plane, we have: $\sum M_z)/x = -M_z(x) + (M_z(x) + dM_z(x)) + \left(T_y(x) + dT_y(x)\right)$

$$\sum F_y = 0 \rightarrow dT_y + q_y dx = 0 \ \rightarrow \boldsymbol{q_y} = -\frac{\boldsymbol{dT_y}}{\boldsymbol{dx}}$$

$$dx + q_y . dx . \frac{dx}{2} = 0$$

$$dM_z(x) + T_y(x)dx = 0 \rightarrow \boldsymbol{T_y} = -\frac{\boldsymbol{dM_z}}{\boldsymbol{dx}} \qquad \rightarrow \boldsymbol{q_y}\ (x) = \frac{\boldsymbol{d^2 M_z(x)}}{\boldsymbol{dx^2}}$$



the displacement field is :

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \ \ et \ \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{v}(\boldsymbol{x})$$

$$\frac{-u(x,y)}{GP} \approx \frac{-u(x,y)}{y} = tg(\theta_z) = \theta_z(x)$$

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{y}\boldsymbol{\theta}_{\boldsymbol{z}}(\boldsymbol{x})$$

the deformation field is :

$$\boldsymbol{\varepsilon}_{xx} = \boldsymbol{\varepsilon}_x = \frac{\partial u}{\partial x} = -\boldsymbol{y}\frac{\boldsymbol{\partial\theta_z(x)}}{\boldsymbol{\partial x}}$$

$$2\boldsymbol{\varepsilon}_{xy} = \boldsymbol{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\boldsymbol{\theta}_z(\boldsymbol{x}) + \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}}$$

the stress field is :

$$\boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_x = E\varepsilon_x = \boldsymbol{E}\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}} = -\boldsymbol{E}\boldsymbol{y}\frac{\boldsymbol{\partial\theta_z(x)}}{\boldsymbol{\partial x}}$$

$$\boldsymbol{\sigma}_{xy} = \boldsymbol{\tau}_{xy} = Gk_y\gamma_{xy} = \boldsymbol{G}\boldsymbol{k}_y\left(-\boldsymbol{\theta}_z(\boldsymbol{x}) + \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}}\right)$$

The internal forces are:

$$\boldsymbol{N} = \int_A \boldsymbol{\sigma}_x \boldsymbol{dA} = -\boldsymbol{E}\frac{\boldsymbol{\partial\theta_z(x)}}{\boldsymbol{\partial x}}\int_A \boldsymbol{y}\boldsymbol{dA} = \boldsymbol{0}$$

$$\boldsymbol{T}_y = \int_A \boldsymbol{\tau}_{xy}\boldsymbol{dA} = \boldsymbol{GA}\boldsymbol{k}_y\left(\frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}} - \boldsymbol{\theta}_z(\boldsymbol{x})\right)$$

$$\boldsymbol{M}_z\vec{\boldsymbol{k}} = \int_A \boldsymbol{y}\vec{\boldsymbol{j}}\wedge\boldsymbol{\sigma}_x\boldsymbol{dA}\vec{\boldsymbol{i}} \rightarrow \boldsymbol{M}_z = \int_A -y\sigma_x dA = E\frac{\partial\theta_z(x)}{\partial x}\int_A y^2 dA = \boldsymbol{E}\boldsymbol{I}_z\frac{\boldsymbol{\partial\theta_z(x)}}{\boldsymbol{\partial x}}$$

This model is the general model known as the ***Timoshenko model***.

If we accept the ***Navier-Bernoulli*** *hypothesis "The cross section remains normal to the deformation of the middle fiber"; Then we neglect the deformation due to shear force and obtain* ***the Bernoulli model***:

$$\boldsymbol{\gamma}_{xy} = -\boldsymbol{\theta}_z(\boldsymbol{x}) + \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}} = \boldsymbol{0} \ \rightarrow \boldsymbol{\theta}_z(\boldsymbol{x}) = \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial x}} \rightarrow \boldsymbol{\varepsilon}_x = -\boldsymbol{y}\frac{\boldsymbol{\partial^2 v}}{\boldsymbol{\partial x^2}} = -\boldsymbol{y}\frac{\boldsymbol{d^2v(x)}}{\boldsymbol{dx^2}}$$

In this case :

$$\boldsymbol{M}_z = \boldsymbol{E}\boldsymbol{I}_z\frac{\boldsymbol{\partial\theta_z(x)}}{\boldsymbol{\partial x}} = \boldsymbol{E}\boldsymbol{I}_z\frac{\boldsymbol{d^2v(x)}}{\boldsymbol{dx^2}}; \quad \boldsymbol{T}_y = -\frac{\boldsymbol{dM(x)}}{\boldsymbol{dx}} = -\boldsymbol{E}\boldsymbol{I}_z\frac{\boldsymbol{d^3v(x)}}{\boldsymbol{dx^3}}$$

The local equilibrium equation is then written as:

$$\boldsymbol{q}_y(\boldsymbol{x}) = \frac{\boldsymbol{d^2M_z(x)}}{\boldsymbol{dx^2}} = \frac{\boldsymbol{d^2}}{\boldsymbol{dx^2}}\left(\boldsymbol{E}\boldsymbol{I}_z\frac{\boldsymbol{d^2v(x)}}{\boldsymbol{dx^2}}\right) \ \ ou \ \ \frac{\boldsymbol{d^2}}{\boldsymbol{dx^2}}\left(\boldsymbol{EI}\frac{\boldsymbol{d^2v(x)}}{\boldsymbol{dx^2}}\right) - \boldsymbol{q}_y(\boldsymbol{x}) = \boldsymbol{0}$$

For a beam with a constant cross-section:

$$\boldsymbol{EI}\frac{\boldsymbol{d^4v(x)}}{\boldsymbol{dx^4}} - \boldsymbol{q}_y(\boldsymbol{x}) = \boldsymbol{0}$$

For bending in the x-z plane and for the Bernoulli model:

$$q_z(x) = -\frac{dT_z(x)}{dx}\ ; T_z(x) = \frac{dM_y(x)}{dx} \rightarrow q_z(x) = -\frac{d^2M_y(x)}{dx^2};\ \sigma_x = \frac{M_y z(x)}{I_y};$$

$$\theta_y(x) = -dw(x)/dx;\ \ u(x,z) = z\theta y(x) = -zdw(x)/dx;$$

$$\varepsilon_x = \frac{\partial u(x)}{\partial x} = -z\frac{\partial \theta_y(x)}{\partial x} = -z\frac{d^2w(x)}{dx^2}$$

$$M_y(x) = -EI\frac{d^2w(x)}{dx^2}\ ;\ \ T_z(x) = EI\frac{d^3w(x)}{dx^3}$$

The local equilibrium equation is:

$$q_z(x) = -\frac{d^2M_y(x)}{dx^2} = -\frac{d^2}{dx^2}\left(-EI\frac{d^2w(x)}{dx^2}\right) \rightarrow \frac{d^2}{dx^2}\left(EI\frac{d^2w(x)}{dx^2}\right) - q_z(x) = 0$$

For a beam with a constant cross-section:

$$EI\frac{d^4w(x)}{dx^4} - q_z(x) = 0$$

2.3.3 Approximation of the displacement field (Shape function) :

For bending in the x-y plane we have approximated the displacement field in the form below:

$$v = [N]\{d\}$$

$$v = N_1v_1 + N_2\theta_{z1} + N_3v_2 + N_4\theta_{z2}\ = N_1\,d_1 + N_2d_2 + N_3d_3 + N_4d_4$$

To make the writing easier, let's voluntarily omit the index z in the writing of θ_z *by simply writing it* θ

$$v = N_1v_1 + N_2\theta_1 + N_3v_2 + N_4\theta_2$$

where:

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3};\ N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2};\ \ N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3};\ N_4 = \frac{-x^2}{L} + \frac{x^3}{L^2}$$

For the case of bending in the x-z plane, we can follow the same procedure to determine the shape functions, but while calculating these, we take the opportunity to present another approximation technique:

Hermite interpolation approximation

The field will be directly approximated as a function of the degrees of freedom:

$$w(x) = N_1w_1 + N_2\theta_1 + N_3w_2 + N_4\theta_2$$
$$= \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \rfloor^T$$

$$\theta(x) = -\frac{dw}{dx} = -\left(\frac{dN_1}{dx}w_1 + \frac{dN_2}{dx}\theta_1 + \frac{dN_3}{dx}w_2 + \frac{dN_4}{dx}\theta_2\right)$$

$$\theta(x) = \lfloor -N_1' \quad -N_2' \quad -N_3' \quad -N_4' \rfloor \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \rfloor^T$$

Let's approximate each $\boldsymbol{N_i}$ *by a polynomial of the form:*

$$N_i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad \rightarrow \quad N_i{}' = a_1 + 2a_2 x + 3a_3 x^2$$

We have the following boundary conditions.:

Node 1: $x = 0\,; w(0) = w_1 = d_1\,;\ \theta(0) = \theta_1 = d_2$

Node 2: $x = L\,; w(L) = w_2 = d_3;\ \theta(L) = \theta_2 = d_4$

Which gives:

$$N_1(0) = 1;\ N_1{}'(0) = 0; N_1(L) = 0;\ N_1{}'(l) = 0$$

$$N_2(0) = 0;\ N_1{}'(0) = -1; N_2(L) = 0;\ N_2{}'(l) = 0$$

$$N_3(0) = 0;\ N_3{}'(0) = 0; N_3(L) = 1;\ N_3{}'(l) = 0$$

$$N_4(0) = 0;\ N_4{}'(0) = 0; N_4(L) = 0;\ N_4{}'(l) = -1$$

Determination of N_1 *:*

$a_0 = 1 \quad and \quad a_1 = 0$

$a_0 + a_1 L + a_2 L^2 + a_3 L^3 = 0 \quad \rightarrow a_2 L^2 + a_3 L^3 = -1 \ \ldots . (a1)$

$a_1 + 2a_2 L + 3a_3 L^2 = 0 \qquad \rightarrow \ 2a_2 L + 3a_3 L^2 = 0 \ \ldots\ldots \ (\mathrm{b1})$

$$a1 \times (3) + b1 \times (-L) \ \rightarrow \ a_2 \mathrm{L}^2 = -3 \ \rightarrow \ a_2 = \frac{-3}{\mathrm{L}^2}$$

$$a1 \times (2) + b1 \times (-L) \ \rightarrow \ -a_3 \mathrm{L}^3 = -2 \rightarrow \ a_3 = \frac{2}{\mathrm{L}^3}$$

$$\boldsymbol{N_1 = 1 - \frac{3x^2}{\mathrm{L}^2} + \frac{2x^3}{\mathrm{L}^3}}$$

The other functions N_2 ; N_3 ; N_4 *are determined in the same way:*

$$\boldsymbol{N_2 = -x + \frac{2x^2}{L} - \frac{x^3}{L^2}}$$

$$\boldsymbol{N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}}$$

$$\boldsymbol{N_4 = \frac{x^2}{L} - \frac{x^3}{L^2}}$$

2.3.4. Formulation of the element:

Let's consider the general case where the element is subjected, in addition to nodal forces $\{f_n\}$, to loads between nodes: (Loads $\boldsymbol{Q_i}$ and/or moments $\boldsymbol{M_i}$ concentrated at a certain number of points « $\boldsymbol{i}$ », load $\boldsymbol{q(x)}$ and/or moment $\boldsymbol{m(x)}$ distributed over a length $[a\ b]$ and a load f_b per unit volume directed along the vertical y or z axis).

Let's consider the case where the bending occurs in the x-y plane:

2.3.4.1 Formulation by the virtual work principle:

$$\boldsymbol{W^{ext} = W^{int}}$$

$$W^{ext} = \{\boldsymbol{d}\}^{\boldsymbol{T}}\{\boldsymbol{f_n}\} + \sum_{i=1}^{k} v_i{}^T\{Q_i\} + \sum_{i=1}^{p} \theta_i{}^T\{M_i\} + \int_a^b v(x)^T\{\,q(X)\}\,dx + \int_a^b \theta(x)^T\{\,m(X)\}\,dx + \int_V v(x)^T f_b\,dx$$

$$W^{ext} = \{d\}^T\{f_n\} + \sum_{i=1}^{k}([N(X = X_i)]\{d\})^T\{Q_i\} + \sum_{i=1}^{p}([N'(X = X_i)]\{d\})^T\{M_i\} + \int_a^b([N(X)]\{d\})^T\,q(X)\,dx + \int_a^b([N'(X)])^T\,m(X)\,dx + \int_V([N(X)]\{d\})^T f_b\,dx$$

$$\boldsymbol{W^{ext} = \{d\}^T\{f\} = \{d\}^T(\{f_n\} + \{f_{eq}\})}$$

$$\boldsymbol{\{f_{eq}\} = \sum_{i=1}^{k}\left(N(X = X_i)\right)^T\{Q_i\} + \sum_{i=1}^{p}\left(N'(X = X_i)\right)^T\{M_i\}}$$

$$\boldsymbol{+\int_s \left(N(X)\right)^T q(X)\,dx + \int_S \left(N'(X)\right)^T m(X)\,dx + \int_V \left(N(X)\right)^T f_b\,dx}$$

$$W^{int} = \int_V \varepsilon^T\sigma dv = \int_V \varepsilon^T[C]\varepsilon dv = \int_V ([B]\{d\})^T[C]([B]\{d\})dv$$

$$W^{int} = \{d\}^T\int_V [B]^T[C][B]dv.\{d\}$$

$$W^{ext} = W^{int} \to \{d\}^T\{f\} = \{d\}^T\int_V [B]^T[C][B]dv.\{d\}$$

$$\{f\} = [k]\{d\}$$

Where: $$\boldsymbol{[k] = \int_V [B]^T[C][B]\,dv}$$

For this element, and neglecting the effect of shear force (Bernoulli model),

$\boldsymbol{\varepsilon}$ *is reduced to* $\boldsymbol{\varepsilon_x = -y\frac{d^2v}{dx^2}} = -y\frac{d^2}{dx^2}\lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \rfloor^T$

$\varepsilon_x = -y\lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor \{d\} \rightarrow [B] = -y\lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor$

$\boldsymbol{\sigma} = [\boldsymbol{C}]\boldsymbol{\varepsilon}$ *, here,* $\boldsymbol{\sigma}$ *is reduced to* σ_x *and as* $\sigma_x = E\varepsilon_x$ *then* $[\boldsymbol{C}]$ *is reduced to* $\boldsymbol{E}$

$$[k] = \int_V -y\begin{bmatrix} N_1'' \\ N_2'' \\ N_3'' \\ N_4'' \end{bmatrix} . E . (-y\lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor) dv$$

$$[k] = E\int_0^L \begin{bmatrix} N_1'' \\ N_2'' \\ N_3'' \\ N_4'' \end{bmatrix} \lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor dx \int_A y^2 ds$$

$$[k] = EI\int_0^L \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} dx$$

2.3.4.2. Formulation by the weighted residues method:

Problem Statement:

The direct rigidity method based on elasticity equations cannot be easily used outside the domain of structural mechanics, for example, in the case of soil consolidation problems. The weighted residues method allows solving such problems. For a problem governed by a differential equation or partial differential equations over the domain Ω of the form:

$$\boldsymbol{D}(U) = 0$$

with boundary conditions on its boundary Γ of the form:

$$\boldsymbol{C}(U, F) = 0$$

where D and C are differential operators

*Approximating the field **U** by:*

$$\bar{U} = [N]\{d\}$$

We will have:

$$\boldsymbol{D}(\bar{U}) = Residue\ R$$

In order to approach exact solutions, we will strive to minimize the integral of the residues over the concerned domain.

$$\int \boldsymbol{R d\Omega = minimum}$$

We can increase the chances of this approximation by associating a weighting function ψ with the residual function and minimizing their product; these chances increase more if we choose this new function in such a way as to obtain a zero minimum

$$\int \boldsymbol{\psi R d\Omega = 0}$$

We can choose the weighting functions ψ in various ways, with each different choice corresponding to a distinct criterion. One of the interesting criteria is the Galerkin criterion, for which the interpolation functions N_i play the role of the weighting functions

$$\int N_i \boldsymbol{D}(\bar{U}) d\Omega = 0$$

This last equation concerns the interior points of the domain without reference to boundary conditions such as applied forces or imposed displacements. To introduce them, we integrate the above equation by parts until they appear. This results in integral expressions for the domain and for its boundaries.

Application to the present case:

For a beam with a constant cross-section, EI is constant, the differential equation is:

$$EI\frac{d^4v}{dx^4} - q_y = 0$$

$$v(x) = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \{d\}$$

The strong form of the weighted residues is written as:

$$\int_0^L N_i\left(EI\frac{d^4v}{dx^4} - q_y\right)dx = 0 \rightarrow EI\int_0^L N_i\frac{d^4v}{dx^4}dx - \int_0^L N_i q_y dx \rightarrow$$

$$\boldsymbol{EI\int_0^L N_i\frac{d^4v}{dx^4}dx = \int_0^L N_i q_y\, dx}$$

Note that the second term can be written for an element loaded over a length $[a \quad b\,]$ in the form:

$$\int_0^L N_i q_y\, dx = \int_0^a (N_i \times 0)\, dx + \int_a^b N_i q_y\, dx + \int_b^L (N_i \times 0)\, dx = \int_a^b N_i q_y\, dx$$

which is the nodal load equivalent to the distributed load q_y : $\{f_{eq}\}$

By applying integration by parts on the first term, we get:

$$EI\int_0^L N_i \frac{d^4v}{dx^4}dx = EI\left(N_i\frac{d^3v}{dx^3}\right)_0^L - EI\int_0^L N_i' \frac{d^3v}{dx^3}dx$$

$$EI\int_0^L N_i \frac{d^4v}{dx^4}dx = EI\left(N_i\frac{d^3v}{dx^3}\right)_0^L - EI\left(\left(N_i'\frac{d^2v}{dx^2}\right)_0^L - \int_0^L N''_i \frac{d^2v}{dx^2}dx\right)$$

$$EI\int_0^L N_i \frac{d^4v}{dx^4}dx = EI\left(N_i\frac{d^3v}{dx^3}\right)_0^L - EI\left(N_i'\frac{d^2v}{dx^2}{}_i\right)_0^L + EI\int_0^L N''_i \frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_i \frac{d^4v}{dx^4}dx = \left(N_i EI\frac{d^3v}{dx^3}\right)_{x=L} - \left(N_i EI\frac{d^3v}{dx^3}\right)_{x=0} - \left(N_i' EI\frac{d^2v}{dx^2}\right)_{x=L} + \left(N_i' EI\frac{d^2v}{dx^2}\right)_{x=0} + EI\int_0^L N''_i \frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_i \frac{d^4v}{dx^4}dx = -N_i(L)T_{y2} + N_i(0)(-T_{y1}) - N_i^{'}(L)(M_{z2}) + N_i^{'}(0)(-M_{z1}) + EI\int_0^L N'_i \frac{d^2v}{dx^2}dx$$

For i=1 we have

$$N_1(0) = 1;\ N_1^{'}(0) = 0; N_1(L) = 0;\ N_1^{'}(L) = 0$$

$$EI\int_0^L N_1 \frac{d^4v}{dx^4}dx = -N_1(L)T_{y2} - N_1(0)T_{y1} - N_1^{'}(L)M_{z2} - N_1^{'}(0)M_{z1} + EI\int_0^L N'_1 \frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_1 \frac{d^4v}{dx^4}dx = -T_{y1} + EI\int_0^L N'_1 \frac{d^2v}{dx^2}dx$$

For i=2 we have

$$N_2(0) = 0;\ N_2^{'}(0) = 1; N_2(L) = 0;\ N_2^{'}(L) = 0$$

$$EI\int_0^L N_2 \frac{d^4v}{dx^4}dx = -N_2(L)T_{y2} - N_2(0)T_{y1} - N_2^{'}(L)M_{z2} - N_2^{'}(0)M_{z1} + EI\int_0^L N'_2 \frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_2 \frac{d^4v}{dx^4}dx = -M_{z1} + EI\int_0^L N'_2 \frac{d^2v}{dx^2}dx$$

For i=3 we have

$$N_3(0)=0;\ N_3^{'}(0)=0; N_3(L)=1;\ N_3^{'}(L)=0$$

$$EI\int_0^L N_3\frac{d^4v}{dx^4}dx=-N_3(L)T_{y2}-N_3(0)T_{y1}-N_3^{'}(L)M_{z2}-N_3^{'}(0)M_{z1}+EI\int_0^L N''_3\frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_2\frac{d^4v}{dx^4}dx=-T_{y2}+EI\int_0^L N''_3\frac{d^2v}{dx^2}dx$$

For i=4 we have

$$N_4(0)=0;\ N_4^{'}(0)=0; N_4(L)=0;\ N_4^{'}(L)=1$$

$$EI\int_0^L N_4\frac{d^4v}{dx^4}dx=N_4(L)T_{y2}-N_4(0)T_{y1}-N_4^{'}(L)M_{z2}+N_4^{'}(0)M_{z1}+EI\int_0^L N''_4\frac{d^2v}{dx^2}dx$$

$$EI\int_0^L N_4\frac{d^4v}{dx^4}dx=-M_{z2}+EI\int_0^L N''_4\frac{d^2v}{dx^2}dx$$

$$\int_0^L [N]^T EI\frac{d^4v}{dx^4}dx=-\begin{Bmatrix}T_{y1}\\M_{z1}\\T_{y2}\\M_{z2}\end{Bmatrix}+EI\int_0^L\left(\frac{d^2[N]}{dx^2}\right)^T\frac{d^2[N]\{d\}}{dx^2}dx=\int_0^L[N]^T q_y\,dx$$

We notice that $\begin{Bmatrix}T_{y1}\\M_{z1}\\T_{y2}\\M_{z2}\end{Bmatrix}$ *is the nodal forces vector* $\{f_n\}$ *and* $\int_0^L[N]^T q_y\,dx$ *is the equivalent forces vector*

Finally, we will have:

$$\{f_n\}+\{f_{eq}\}=(EI\int_0^L[N]''^T[N]''dx)\{d\}=$$

$$\{f\}=[k]\{d\}\ \text{où}\ \{f\}=\{f_n\}+\{f_{eq}\}$$

$$\text{and } [k]=EI\int_0^L[N]''^T[N]''dx=EI\int_0^L\begin{bmatrix}N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4''\\N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4''\\N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4''\\N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4''\end{bmatrix}dx$$

We thus find the same matrix determined by the direct method.

Formulation by variational methods:

Problem statement:

Consider a differential equation of the form: $\boldsymbol{D}\,(\boldsymbol{u}) = \boldsymbol{0}$.

For this differential equation, if a functional $\boldsymbol{\pi}$ *exists such that:*

$$\boldsymbol{\pi} = \int_a^b \boldsymbol{f}[\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}'(\boldsymbol{x})]\boldsymbol{dx}$$

Then, among the admissible functions $\boldsymbol{u}(\boldsymbol{x})$, *the one that gives the closest solution to the equation* $\boldsymbol{D}(\boldsymbol{u}) = \boldsymbol{0}$ *is the one that gives the functional* $\boldsymbol{\pi}$ *in the interval* $[\boldsymbol{a}, \boldsymbol{b}]$ *the extreme value (maximum or minimum) for a given functional* $\boldsymbol{f}$

The function $\mathbf{u}(\boldsymbol{x})$ *that maximizes or minimizes* $\boldsymbol{f}$ *is called the stationary function. In variational calculus, it can be shown that the stationary function*

$$\boldsymbol{f}[\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}'(\boldsymbol{x})]$$

must satisfy the Euler-Lagrange equation

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} - \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}'}\right) = \boldsymbol{0}$$

In the variational approach, the nodal values of $\mathbf{u}_i$ *must be found to make the functional* $\boldsymbol{\pi}(\boldsymbol{u})$ *stationary such that:*

$$\boldsymbol{\delta\pi}(\boldsymbol{u}) = \sum_{k=1}^{n} \frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{u}_k} \boldsymbol{\delta u}_k = \boldsymbol{0}$$

where $\boldsymbol{n}$ *is the number of discrete solutions* $\mathbf{u}_i$ *assigned to the domain. From the previous equation, it follows that*

$$\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{u}_k} = \boldsymbol{0}, \qquad \boldsymbol{k} = \boldsymbol{1}, \boldsymbol{2}, \ldots\ldots\ldots; \boldsymbol{n}$$

Application to the present case:

Here, the functional $\boldsymbol{\pi}$ *is the potential energy*

$$\boldsymbol{E} = \boldsymbol{U} - \boldsymbol{P}$$

$$\boldsymbol{U} = \int_V \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{\varepsilon}^{\boldsymbol{T}} \boldsymbol{\sigma} \boldsymbol{dV}$$

$$P = \{\boldsymbol{d}\}^{\boldsymbol{T}}\{\boldsymbol{f}_{\boldsymbol{n}}\} + \sum_{i=1}^{k} v_i{}^T\{Q_i\} + \sum_{i=1}^{n} \theta_i{}^T\{M_i\} + \int_0^L v(x)^T\{\,q(X)\}\,dx + \int_0^L \theta(x)^T\{\,m(X)\}\,dx$$
$$+ \int_V v(x)^T f_b\,dx$$

For the beam element - bending bar element - (Bernoulli model):

$$\varepsilon = \varepsilon_x = u' \quad ; \quad \sigma = \sigma_x = E\varepsilon_x = Eu'$$

In the x-y plane, we have:

$$v = [N]\{d\} = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \quad ; \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \rfloor = \lfloor v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \rfloor$$

$$\theta = [N]\{d\} = N_1 d_1 + N_2 d_2 + N_3 d_3 + N_4 d_4 \quad ; \lfloor d_1 \quad d_2 \quad d_3 \quad d_4 \rfloor = \lfloor v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \rfloor$$

$$U = \int_V \frac{1}{2}\varepsilon_x^T \sigma_x dV = \int_V \frac{E}{2} u'^T . u' . dV = \frac{E}{2}\int_V \left(-y\frac{d^2v}{dx^2}\right)^T \left(-y\frac{d^2v}{dx^2}\right) dV$$

$$U = \frac{E}{2}\int_V y^2 \left(\frac{d^2v}{dx^2}\right)^T \left(\frac{d^2v}{dx^2}\right) dV$$

$$U = \frac{EI_z}{2}\int_0^L (\lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor \{d\})^T (\lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor \{d\})\, dx$$

$$U = \frac{EI_z}{2}\int_0^L \{d\}^T \lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor^T \lfloor N_1'' \quad N_2'' \quad N_3'' \quad N_4'' \rfloor \{d\}\, dx$$

$$U = \frac{EI_z}{2}\int_0^L (N_1'' d_1 + N_2'' d_2 + N_3'' d_3 + N_4'' d_4)^2\, dx$$

$$P = \{d\}^T\{f_n\} + \sum_{i=1}^{k} ([N(X = X_i)]\{d\})^T\{Q_i\} + \sum_{i=1}^{p} ([N'(X = X_i)]\{d\})^T\{M_i\} +$$

$$\int_s ([N(X)]\{d\})^T q(X)\, dx + \int_S ([N'(X)])^T m(X)\, dx + \int_V ([N(X)]\{d\})^T f_b\, dx$$

$$P = \{d\}^T \left(\begin{array}{c} \{f_n\} + \int_0^L [N]^T\{q(X)\}\, dx + \int_0^L [N']^T\{m(X)\}\, dx + \sum_{i=1}^{k} [N(x = x_i)]^T\{Q_i\} + \\ \sum_{i=1}^{n} [N'(x = x_i]^T\{M_i\} + \int_V [N(X)]^T f_b dx \end{array} \right)$$

The minimization of potential energy is written as:

$$\frac{\partial E}{\partial d_i} = \frac{\partial U}{\partial d_i} + \frac{\partial P}{\partial d_i} = 0 \quad \rightarrow \quad \frac{\partial U}{\partial d_i} = -\frac{\partial P}{\partial d_i}$$

$$\frac{\partial U}{\partial d_1} = EI_z \int_0^L N_1'' (N_1'' d_1 + N_2'' d_2 + N_3'' d_3 + N_4'' d_4) dx$$

$$\frac{\partial U}{\partial d_1} = EI_z \int_0^L \lfloor N_1'' N_1'' \quad N_1'' N_2'' \quad N_1'' N_3'' \quad N_1'' N_4'' \rfloor \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

In the same way, we find:

$$\frac{\partial U}{\partial d_2} = EI_z \int_0^L \lfloor N_2''N_1'' \quad N_2''N_2'' \quad N_2''N_3'' \quad N_2''N_4'' \rfloor \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\frac{\partial U}{\partial d_3} = EI_z \int_0^L \lfloor N_3''N_1'' \quad N_3''N_2'' \quad N_3''N_3'' \quad N_3''N_4'' \rfloor \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\frac{\partial U}{\partial d_1} = EI_z \int_0^L \lfloor N_4''N_1'' \quad N_4''N_2'' \quad N_4''N_3'' \quad N_4''N_4'' \rfloor \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\frac{\partial U}{\lfloor \partial d \rfloor} = EI_z \int_0^L \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} \{d\}$$

$$-\frac{\partial P}{\partial d} = \{f_n\} + \int_0^L [N]^T\{\, q(X)\}\, dx + \int_0^L [N']^T\{\, m(X)\}\, dx + \sum_{i=1}^k [N(x = x_i)]^T \{Q_i\} +$$

$$\sum_{i=1}^{n} [N'(x = x_i]^T \{M_i\} + \int_V \; [N(X)]^T f_b dx$$

$$-\frac{\partial P}{\lfloor \partial d \rfloor} = \{f_n\} + \{f_{eq}\} = \{f\}$$

$$\frac{\partial U}{[\partial d[} = -\frac{\partial P}{\lfloor \partial d \rfloor} \rightarrow \{f\} = EI_z \int_0^L \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} \{d\} = [k]\{d\}$$

$$[k] = EI_z \int_0^L \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix}$$

To evaluate the terms of $[k]$ ***,*** *let's remember that:*

$$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3};\ N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2};\quad N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3};\ N_4 = \frac{-x^2}{L} + \frac{x^3}{L^2}$$

Then

$$N_1'' = -\frac{6}{L^2} + \frac{12x}{L^3};\ N_2'' = -\frac{4}{L} + \frac{6x}{L^2};\quad N_3'' = \frac{6}{L^2} - \frac{12x}{L^3};\ N_4'' = -\frac{2}{L} + \frac{6x}{L^2}$$

$$k_{11} = EI_z\int_0^L N_1''N_1''dx = EI\int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)\left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)dx$$

$$k_{11} = EI_z\int_0^L \left(\frac{36}{L^4} - \frac{144x}{L^5} + \frac{144x^2}{L^6}\right)dx = EI_z\left[\frac{36x}{L^4} - \frac{72x^2}{L^5} + \frac{48x^3}{L^6}\right]_0^L = \frac{12EI_z}{L^3}$$

$$k_{12} = EI_z\int_0^L N_1''N_2''dx = EI_z\int_0^L \left(-\frac{6}{L^2} + \frac{12x}{L^3}\right)\left(-\frac{4}{L} + \frac{6x}{L^2}\right)dx$$

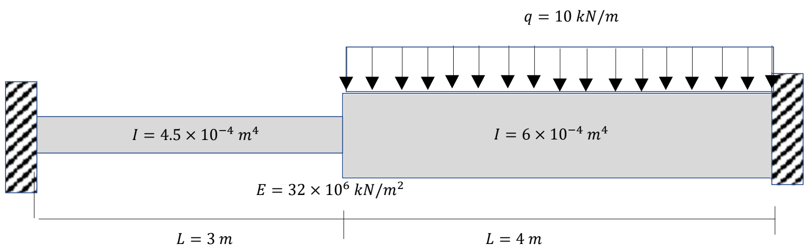
$$k_{12} = EI_z\int_0^L \left(\frac{24}{L^3} - \frac{84x}{L^4} + \frac{72x^2}{L^5}\right)dx = EI_z\left[\frac{24x}{L^3} - \frac{42x^2}{L^4} + \frac{24x^3}{L^5}\right]_0^L = \frac{6EI_z}{L^2}$$

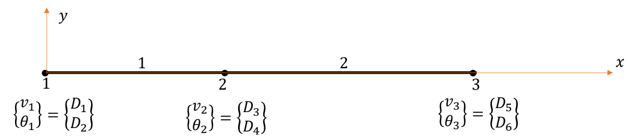
$$[k] = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

For the bending in the x-z plane, a similar calculation gives:

$$[k] = \frac{EI_y}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}$$

2.3.5. Study of a beam structure:





For a beam (continuous beam) subdivided into elements (spans), the local axes of the elements and the global axes of the beam coincide (taking the direction of the global x-axis as that of the longitudinal axis of the beam

The transformation matrices of the elements are therefore identity matrices, and consequently, the degrees of freedom, nodal forces, and stiffness matrices in the local and global reference coordinate are identical

2.3.5.1. Calculation of the stiffness matrices of the elements:

Element 1: $L = 3\ m\ ;\ I = 4.5 \times 10^{-4} m^{-4}\ ;\ \ E = 32 \times 10^6\ kN/m^2$

$$[k]^1 = [K]^1 = \frac{32 \times 10^6 \times 4.5 \times 10^{-4}}{3^3} \begin{bmatrix} 12 & 18 & -12 & 18 \\ 18 & 36 & -18 & 18 \\ -12 & -18 & 12 & -18 \\ 18 & 18 & -18 & 36 \end{bmatrix}$$

$$[k]^1 = [K]^1 = \begin{bmatrix} 6400 & 9600 & -6400 & 9600 \\ 9600 & 19200 & -9600 & 9600 \\ -6400 & -9600 & 6400 & -9600 \\ 9600 & 9600 & -9600 & 19200 \end{bmatrix}$$

Element 2 : $L = 4\ m\ ;\ I = 6 \times 10^{-4} m^{-4}\ ;\ \ E = 32 \times 10^6\ kN/m^2$

$$[k]^2 = [K]^2 = \frac{32 \times 10^6 \times 6 \times 10^{-4}}{4^3} \begin{bmatrix} 12 & 24 & -12 & 24 \\ 24 & 64 & -24 & 32 \\ -12 & -24 & 12 & -24 \\ 24 & 32 & -24 & 64 \end{bmatrix}$$

$$[k]^2 = [K]^2 = \begin{bmatrix} 3600 & 7200 & -3600 & 7200 \\ 7200 & 19200 & -7200 & 9600 \\ -3600 & -7200 & 3600 & -7200 \\ 7200 & 9600 & -7200 & 19200 \end{bmatrix}$$

2.3.5.2. Calculation of the equivalent loads:

The nodal loads equivalent to the uniformly distributed load on element 2 (the right segment) are:

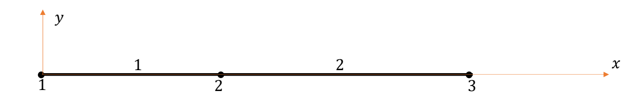
$$\{f_{eq}\} = \int_{\Omega} [N]^T . q(X). d\Omega = \int_0^L [N]^T . q(x) dx$$

$$\{f_{eq}\}^2 = \int_0^L \begin{bmatrix} 1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3} \\ x-\frac{2x^2}{L}+\frac{x^3}{L^2} \\ \frac{3x^2}{L^2}-\frac{2x^3}{L^3} \\ \frac{-x^2}{L}+\frac{x^3}{L^2} \end{bmatrix} \times (-10).dx = -10 \times \begin{bmatrix} x-\frac{x^3}{L^2}+\frac{x^4}{2L^3} \\ \frac{x^2}{2}-\frac{2x^3}{3L}+\frac{x^4}{4L^2} \\ \frac{3x^3}{2L^2}-\frac{x^4}{2L^3} \\ -\frac{x^3}{3L}+\frac{x^4}{4L^2} \end{bmatrix}_0^L = \begin{Bmatrix} -20 \\ -13.33 \\ -20 \\ 13.33 \end{Bmatrix}$$

2.3.5.3. Calculation of the global stiffness matrix:

The continuity of displacements gives:

$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} ; \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} ; \quad \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}^2 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^2 = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$



$\begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}$ $\quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix}$ $\quad \begin{Bmatrix} v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} D_5 \\ D_6 \end{Bmatrix}$

The assembly gives:

$[K] =$

6400	9600	−6400	9600		
9600	19200	−9600	9600		
−6400	−9600	6400 3600	−9600 7200	−3600	7200
9600	9600	−9600 7200	19200 19200	−7200	9600
		−3600	−7200	3600	−7200
		7200	9600	−7200	19200

The global system is thus:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} 6400 & 9600 & -6400 & 9600 & 0 & 0 \\ 9600 & 19200 & -9600 & 9600 & 0 & 0 \\ -6400 & -9600 & 10000 & -2400 & -3600 & 7200 \\ 9600 & 9600 & -2400 & 38400 & -7200 & 9600 \\ 0 & 0 & -3600 & -7200 & 3600 & -7200 \\ 0 & 0 & 7200 & 9600 & -7200 & 19200 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

Another direct method of assembly is the ***congruence transformation method****:*

If we arrange the equations of the elements as follows

$$\begin{Bmatrix} \{f\}^1 \\ \{f\}^2 \\ \vdots \\ \{f\}^m \end{Bmatrix} = \begin{bmatrix} [K]^1 & & & \\ & [K]^2 & & \\ & & \ddots & \\ & & & [K]^m \end{bmatrix} \begin{Bmatrix} \{D\}^1 \\ \{D\}^2 \\ \vdots \\ \{D\}^m \end{Bmatrix}$$

Or :

$$\{F_e\} = [K_e]\{D_e\}$$

$[K_e]$ *: diagonal matrix of element stiffness submatrices, called the unassembled global stiffness matrix.*

$\{F_e\}$ *: vector whose terms are the force vectors of the elements called the unassembled global force vector..*

$\{D_e\}$ *: vector whose terms are the vectors of the degrees of freedom of the elements and are called the unassembled global vector of degrees of freedom.*

To perform the assembly of the elements, it is sufficient to consider the continuity of displacements at the nodes of the structure, which is expressed in the form:

$$\{D_e\} = [C_e]\{D\}$$

Where $\{D\}$*enumerates the global displacements at the nodes of the structure and* $[C_e]$ *is called the global kinematic matrix or connection matrix,*

Similarly:

$$\{F_e\} = [B_e]\{F\}$$

Where $\{F\}$ *enumerates the external forces at the nodes of the structure, and* $[B_e]$ *is called the global static matrix*

The virtual work of the external forces is:

$$W^{ext} = \{D\}^T\{F\} = \{D\}^T[B_e]^{-1}\{F_e\}$$

The work of the internal forces is:

$$W^{int} = \{D_e\}^T\{\{F_e\}\} = ([C_e]\{D\})^T\{F_e\} = \{D\}^T[C_e]^T\{F_e\}$$

$$[B_e]^{-1} = [C_e]^T$$

Noting that $[C_e]$ *is orthogonal, thus* $[C_e]^T = [C_e]^{-1}$

Which gives:

$$[B_e] = [C_e]$$

We have:

$$\{\{F_e\}\} = [B_e]\{F\} = [C_e]\{F\} = [K_e]\{D_e\} = [K_e][C_e]\{D\} \rightarrow \{F\} = [C_e]^{-1}[K_e][C_e]\{D\}$$

$$\rightarrow \{F\} = [K]\{D\} \quad where$$

$$[K] = [C_e]^{-1}[K_e][C_e]$$

As $[C_e]$ is orthogonal $\quad [C_e]^{-1}=[C_e]^T$

$$[K] = [C_e]^T[K_e][C_e]$$

The construction of the matrix $[C_e]$ is simple. If the elements i, j, and k share a degree of freedom D_p , then the compatibility condition for displacements implies:

$$D_p = {D_p}^i = {D_p}^j = {D_p}^k$$

This equation generates a column of the matrix $[C^e]$: unit values for the rows corresponding to ${D_p}^i = {D_p}^j = {D_p}^k$ and zeros elsewhere. Such a matrix populated only with elements equal to zero or one is called a Boolean matrix and is well-suited for designing highly efficient algorithms.

For our example:

$$\{D_e\} = \begin{Bmatrix} \{D\}^1 \\ \{D\}^2 \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^1 \\ \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}^2 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}; \ \{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

Thus

$$[C_e] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_e]=\begin{bmatrix} 6400 & 9600 & -6400 & 9600 & & & & \\ 9600 & 19200 & -9600 & 9600 & & & & \\ -6400 & -9600 & 6400 & -9600 & & & & \\ 9600 & 9600 & -9600 & 19200 & & & & \\ & & & & 3600 & 7200 & -3600 & 7200 \\ & & & & 7200 & 19200 & -7200 & 9600 \\ & & & & -3600 & -7200 & 3600 & -7200 \\ & & & & 7200 & 9600 & -7200 & 19200 \end{bmatrix}$$

The double matrix product: $[C_e]^{-1}[K^e][C_e]$ *performed by MATLAB gives the same matrix K determined previously*

:

$$\{K\}=\begin{bmatrix} \mathbf{6400} & \mathbf{9600} & \mathbf{-6400} & \mathbf{9600} & \mathbf{0} & \mathbf{0} \\ \mathbf{9600} & \mathbf{19200} & \mathbf{-9600} & \mathbf{9600} & \mathbf{0} & \mathbf{0} \\ \mathbf{-6400} & \mathbf{-9600} & \mathbf{10000} & \mathbf{-2400} & \mathbf{-3600} & \mathbf{7200} \\ \mathbf{9600} & \mathbf{9600} & \mathbf{-2400} & \mathbf{38400} & \mathbf{-7200} & \mathbf{9600} \\ \mathbf{0} & \mathbf{0} & \mathbf{-3600} & \mathbf{-7200} & \mathbf{3600} & \mathbf{-7200} \\ \mathbf{0} & \mathbf{0} & \mathbf{7200} & \mathbf{9600} & \mathbf{-7200} & \mathbf{19200} \end{bmatrix}$$

4.2.3.5.4. Introduction of the boundary conditions:

$$\boldsymbol{F_1=R_{y1}\,;\ F_2=M_{e1}\ ;\ F_3=0+f_{eq1}^2=-20\ kN\,;\ F_4=0+f_{eq2}^2=-13.33\ ;}$$

$$\boldsymbol{F_5=R_{y3}+f_{eq3}^2=R_{y3}-20\,;\ F_6=M_{e3}+f_{eq4}^2=M_{e3}+13.33}$$

$$\boldsymbol{D_1=D_2=0;D_3=?;\ D_4=?;D_5=D_6=0}$$

$$\begin{Bmatrix} R_{y1} \\ M_{e1} \\ -20 \\ -13.33 \\ \boldsymbol{R_{y3}}-20 \\ \boldsymbol{M_{e3}}+13.33 \end{Bmatrix}=\begin{bmatrix} 6400 & 9600 & -6400 & 9600 & 0 & 0 \\ 9600 & 19200 & -9600 & 9600 & 0 & 0 \\ -6400 & -9600 & 10000 & -2400 & -3600 & 7200 \\ 9600 & 9600 & -2400 & 38400 & -7200 & 9600 \\ 0 & 0 & -3600 & -7200 & 3600 & -7200 \\ 0 & 0 & 7200 & 9600 & -7200 & 19200 \end{bmatrix}\begin{Bmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \\ \boldsymbol{D_3} \\ \boldsymbol{D_4} \\ \boldsymbol{D_5} \\ \boldsymbol{D_6} \end{Bmatrix}$$

After the introduction of boundary conditions, this system reduces to a reduced system:

$$\{\boldsymbol{F}^*\} = [\boldsymbol{K}^*]\{\boldsymbol{D}^*\} \rightarrow \begin{Bmatrix} -20 \\ -13.33 \end{Bmatrix} = \begin{bmatrix} 10000 & -2400 \\ -2400 & 38400 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix}$$

$$\boldsymbol{D_3 = v_2 = -\,0.002115\ m}$$

$$\boldsymbol{D_4 = \theta_2 = -0.0004794\ rd}$$

Note: *For manual calculation and to directly determine the reduced system, boundary conditions are introduced at the element level.*

For example, using the congruence transformation method, we will have:

$$[k^*]^1 = \begin{bmatrix} 6400 & -9600 \\ -9600 & 19200 \end{bmatrix};\ \{D^*\}^1 = \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix} \quad [k^*]^2 = \begin{bmatrix} 3600 & 7200 \\ 7200 & 19200 \end{bmatrix};\ \{D^*\}^2 = \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix}$$

$$[K_e^*] = \begin{bmatrix} 6400 & -9600 & & \\ -9600 & 19200 & & \\ & & 3600 & 7200 \\ & & 7200 & 19200 \end{bmatrix}$$

$$\{D_e^*\} = \begin{Bmatrix} D_3 \\ D_4 \\ D_3 \\ D_4 \end{Bmatrix}, \qquad \{D^*\} = \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix};\ \{F^*\} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix},$$

$$\{C_e^*\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\boldsymbol{K}^*] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6400 & -9600 & 0 & 0 \\ -9600 & 19200 & 0 & 0 \\ 0 & 0 & 3600 & 7200 \\ 0 & 0 & 7200 & 19200 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 6400 & -9600 & 3600 & 7200 \\ -9600 & 19200 & 7200 & 19200 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{10000} & \mathbf{-2400} \\ \mathbf{-2400} & \mathbf{38400} \end{bmatrix}$$

4.2.3.5.5. Determination of the support reactions and the nodal forces of the elements:

To calculate the support reactions, we use the global system:

$$\begin{Bmatrix} R_{y1} \\ M_{e1} \\ -20 \\ -13.33 \\ R_{y3}+(-20) \\ M_{e3}+(13.33) \end{Bmatrix} = \begin{bmatrix} 6400 & 9600 & -6400 & 9600 & 0 & 0 \\ 9600 & 19200 & -9600 & 9600 & 0 & 0 \\ -6400 & -9600 & 10000 & -2400 & -3600 & 7200 \\ 9600 & 9600 & -2400 & 38400 & -7200 & 9600 \\ 0 & 0 & -3600 & -7200 & 3600 & -7200 \\ 0 & 0 & 7200 & 9600 & -7200 & 19200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.002115 \\ -0.0004794 \\ 0 \\ 0 \end{Bmatrix}$$

$$R_{y1} = -6400 \times (-0.002115) + 9600 \times (-0.0004794) = 8.93\ kN,$$

$$M_{e1} = -9600 \times (-0.002115) + 9600 \times (-0.0004794) = 15.70\ kNm,$$

$$R_{y3} - 20 = -3600 \times (-0.002115) - 7200 \times (-0.0004794) \rightarrow R_{y2} = 31.07\ kN,$$

$$M_{e3} + 13.33 = 7200 \times (-0.002115) + 9600 \times (-0.0004794) \rightarrow M_{e2} = -31.16\ kNm,$$

To calculate the nodal forces of an element 'i,' we first determine its nodal displacements in global coordinates and then in local coordinates. Then, from the system:

$$\{f\}^i = [k]^i\{d\}^i \quad avec \quad \{f\}^i = \{f_n\}^i + \{f_{eq}\}^i$$

we calculate the nodal forces

$$\{f_n\}^i = [k]^i\{d\}^i - \{f_{eq}\}^i$$

$$\{d\}^1 = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}^1 = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.002115 \\ -0.0004794 \end{Bmatrix};\ \{d\}^2 = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}^2 = \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -0.002115 \\ -0.0004794 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{f_n\}^1 = \begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^1 = \{f\}^1 - \{f_{eq}\}^1 = \{f\}^1 - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [k]^1\{d\}^1$$

$$\begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^1 = \begin{bmatrix} 6400 & 9600 & -6400 & 9600 \\ 9600 & 19200 & -9600 & 9600 \\ -6400 & -9600 & 6400 & -9600 \\ 9600 & 9600 & -9600 & 19200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.002115 \\ -0.0004794 \end{Bmatrix} = \begin{Bmatrix} \mathbf{8.94} \\ \mathbf{15.7} \\ \mathbf{-8.94} \\ \mathbf{11.1} \end{Bmatrix}$$

$$\{f_n\}^2 = \begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^2 = \{f\}^2 - \{f_{eq}\}^2 = \{f\}^2 - \{\quad\} = [k]^2\{d\}^2 - \begin{Bmatrix} -20 \\ -13.33 \\ -20 \\ 13.33 \end{Bmatrix}$$

$$\begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^2 = \begin{bmatrix} 3600 & 7200 & -3600 & 7200 \\ 7200 & 19200 & -7200 & 9600 \\ -3600 & -7200 & 3600 & -7200 \\ 7200 & 9600 & -7200 & 19200 \end{bmatrix} \begin{Bmatrix} -0.002115 \\ -0.0004794 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -20 \\ -13.33 \\ -20 \\ 13.33 \end{Bmatrix} = \begin{Bmatrix} \mathbf{8.94} \\ \mathbf{-11.10} \\ \mathbf{31.07} \\ \mathbf{-33.16} \end{Bmatrix}$$

2.3.6. *Hinged elements:*

a- *Element hinged at its left node (Hinge-Rigid Element):*

for this element, the moment is zero at this node, so in the element system:

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$f_2 = M_{e1} = \frac{EI_z}{L^3}(6Ld_1 + 4L^2 d_2 - 6Ld_3 + 2L^2 d_4) = 0 \ \rightarrow$$

$$\boldsymbol{d_2 = -\frac{3}{2L}d_1 + \frac{3}{2L}d_3 - \frac{1}{2}d_4}$$

$$\boldsymbol{\theta_1 = -\frac{3}{2L}v_1 + \frac{3}{2L}v_2 - \frac{1}{2}\theta_2}$$

$$\boldsymbol{N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3};\ N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2};\ \ N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3};\ N_4 = \frac{-x^2}{L} + \frac{x^3}{L^2}}$$

$$\boldsymbol{N_{p1} = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} - \frac{3x}{2L} + \frac{3x^2}{L^2} - \frac{3x^3}{2L^3} = 1 - \frac{3x}{2L} + \frac{x^3}{2L^3}}$$

$$\boldsymbol{N_{p3} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} + \frac{3x}{2L} - \frac{3x^2}{L^2} + \frac{3x^3}{2L^3} = \frac{3x}{2L} - \frac{x^3}{2L^3}}$$

$$\boldsymbol{N_{p4} = \frac{-x^2}{L} + \frac{x^3}{L^2} - \frac{x}{2} + \frac{x^2}{L} - \frac{x^3}{2L^2} = -\frac{x}{2} + \frac{x^3}{2L^2}}$$

$$\begin{Bmatrix} f_1 \\ 0 \\ f_3 \\ f_4 \end{Bmatrix} = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_1 \\ \boldsymbol{-d_1 + \frac{3}{2L}d_3 - \frac{1}{2}d_4} \\ d_3 \\ d_4 \end{Bmatrix}$$

$$f_1 = \frac{EI_z}{L^3}\left(\left(12 - \boldsymbol{6L\frac{3}{2L}}\right)\boldsymbol{d_1} + \left(-12 + \boldsymbol{6L\frac{3}{2L}}\right)\boldsymbol{d_3} + \left(6L - \boldsymbol{6L\frac{1}{2}}\right)\boldsymbol{d_4}\right)$$

$$f_1 = \frac{3EI_z}{L^3}(\boldsymbol{d_1} - \boldsymbol{d_3} + L\boldsymbol{d_4})$$

$$f_3 = \frac{EI_z}{L^3}\left(\left(-12 + \boldsymbol{6L}\frac{\mathbf{3}}{\mathbf{2L}}\right)\boldsymbol{d_1} + \left(12 - \boldsymbol{6L}\frac{\mathbf{3}}{\mathbf{2L}}\right)\boldsymbol{d_3} + \left(-6L + \boldsymbol{6L}\frac{\mathbf{1}}{\mathbf{2}}\right)\boldsymbol{d_4}\right)$$

$$f_3 = \frac{3EI_z}{L^3}(-\boldsymbol{d_1} + \boldsymbol{d_3} - L\boldsymbol{d_4})$$

$$f_4 = \frac{EI_z}{L^3}\left(\left(6L - 2L^2\frac{\mathbf{3}}{\mathbf{2L}}\right)\boldsymbol{d_1} + \left(-6L + 2L^2\frac{\mathbf{3}}{\mathbf{2L}}\right)\boldsymbol{d_3} + \left(4L^2 - 2L^2\frac{\mathbf{1}}{\mathbf{2}}\right)\boldsymbol{d_4}\right)$$

$$f_4 = \frac{3EI_z}{L^3}(L\boldsymbol{d_1} - L\boldsymbol{d_3} + L^2\boldsymbol{d_4})$$

So, for this element:

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & 0 & -1 & L \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -L \\ L & 0 & -L & L^2 \end{bmatrix}$$

b- Element hinged at its right node (Rigid - Hinge Element):

A similar reasoning gives:

$$\theta_2 = -\frac{3}{2L}v_1 + \frac{3}{2L}v_2 - \frac{1}{2}\theta_1$$

$$\boldsymbol{N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3};\ N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2};\ \ N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3};\ N_4 = \frac{-x^2}{L} + \frac{x^3}{L^2}}$$

$$\boldsymbol{N_{p1} = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{3x^2}{2L^2} - \frac{3x^3}{2L^3} = 1 - \frac{3x^2}{2L^2} + \frac{x^3}{2L^3}}$$

$$\boldsymbol{N_{p2} = x - \frac{2x^2}{L} + \frac{x^3}{L^2} + \frac{x^2}{2L} - \frac{x^3}{2L^2} = x - \frac{3x^2}{2L} + \frac{x^3}{2L^2}}$$

$$\boldsymbol{N_{p3} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} - \frac{3x^2}{2L^2} + \frac{3x^3}{2L^3} = \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}}$$

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & L & -1 & 0 \\ L & L^2 & -L & 0 \\ -1 & -L & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c- ***Element hinged at its two nodes (Hinge-Hing Element):***

$$d_2 = \frac{-d_1}{L} + \frac{d_3}{L};\ d_4 = \frac{-d_1}{L} + \frac{d_3}{L}$$

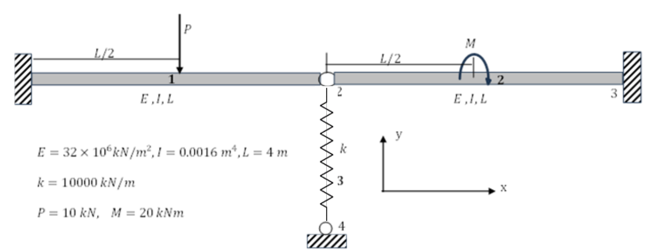
$$\theta_1 = \frac{-v_1 + v_2}{L} = \theta_2$$

$$\boldsymbol{N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3};\ N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2};\ \ N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3};\ N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}}$$

$$\boldsymbol{N_{1p} = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} - \frac{x}{L} + \frac{2x^2}{L^2} - \frac{x^3}{L^3} + \frac{x^2}{L^2} - \frac{x^3}{L^3} = 1 - \frac{x}{L}}$$

$$\boldsymbol{N_{3p} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} + \frac{x}{L} - \frac{2x^2}{L^2} + \frac{x^3}{L^3} - \frac{x^2}{L^2} + \frac{x^3}{L^3} = \frac{x}{L}}$$

$$[k] = \frac{3EI_z}{L^3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



For the beam the global and local coordinate are the same:

Element 1: left span (rigid- Hing)

$$Dof: v_1, \theta_1, v_2, \theta_2$$

$$[k]^1 = [K]^1 = \begin{bmatrix} 2400 & 9600 & -2400 & 0 \\ 9600 & 38400 & -9600 & 0 \\ -2400 & -9600 & 2400 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element 2: right span (Hing- rigid)

$$Dof: v_2, \theta_2, v_3, \theta_3$$

$$[k]^2 = [K]^2 = \begin{bmatrix} 2400 & 0 & -2400 & 9600 \\ 0 & 0 & 0 & 0 \\ -2400 & 0 & 2400 & -9600 \\ 9600 & 0 & -9600 & 38400 \end{bmatrix}$$

Element 3 : Spring

$$Dof: u_2, v_2, u_4, v_4$$

$$l = 0, m = -1 \quad \rightarrow \quad [T]^3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$[k]^3 = \begin{bmatrix} 10000 & -10000 \\ -10000 & 10000 \end{bmatrix}; \quad [K]^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10000 & 0 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & -10000 & 0 & 10000 \end{bmatrix}$$

Equivalent loads:

$$\begin{Bmatrix} f_{eq1} \\ f_{eq2} \\ f_{eq3} \\ f_{eq4} \end{Bmatrix}^1 = \begin{bmatrix} \mathbf{1 - \frac{3x^2}{2L^2} + \frac{x^3}{2L^3}} \\ \mathbf{x - \frac{3x^2}{2L} + \frac{x^3}{2L^2}} \\ \mathbf{\frac{3x^2}{2L^2} - \frac{x^3}{2L^3}} \\ \mathbf{0} \end{bmatrix}_{x=L/2} \times (-10) = \begin{Bmatrix} -6.875 \\ -7.5 \\ -3.125 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} f_{eq1} \\ f_{eq2} \\ f_{eq3} \\ f_{eq4} \end{Bmatrix}^2 = \begin{bmatrix} -\dfrac{3}{2L}+\dfrac{3\boldsymbol{x}^2}{2\boldsymbol{L}^3} \\ 0 \\ \dfrac{3}{2\boldsymbol{L}}-\dfrac{3\boldsymbol{x}^2}{2\boldsymbol{L}^3} \\ -\dfrac{1}{2}+\dfrac{3\boldsymbol{x}^2}{2\boldsymbol{L}^2} \end{bmatrix}_{x=L/2} \times (-20) = \begin{Bmatrix} 5.625 \\ 0 \\ -5.625 \\ 2.5 \end{Bmatrix}$$

In total, we have 3 rotations θ_1 *,* θ_2*,* θ_3*;2 translations* u_2 *,* u_4 *in the x-direction and 4 translations* v_1, v_2, v_3, v_4 *in the y-direction*

$$\{D\} = \lfloor D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9 \rfloor^T = \lfloor v_1, \theta_1, u_2, v_2, \theta_2, v_3, \theta_3, u_4, v_4 \rfloor^T$$

$$\{D\}^1 = \lfloor D_1, D_2, D_4, D_5 \rfloor^T = \lfloor v_1, \theta_1, v_2, \theta_2, \rfloor^T$$

$$\{D\}^2 = \lfloor D_4, D_5, D_6, D_7 \rfloor^T = \lfloor v_2, \theta_2, v_3, \theta_3 \rfloor^T$$

$$\{D\}^3 = \lfloor D_3, D_4, D_8, D_9 \rfloor^T = \lfloor u_2, v_2, u_4, v_4 \rfloor^T$$

$$\{F\} = \lfloor F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9 \rfloor^T$$

$$\{F\} = \lfloor F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9 \rfloor^T$$

$$= \lfloor R_{y1} + f_{eq1}^1, M_{ez1} + f_{eq2}^1, 0, f_{eq3}^1 + f_{eq1}^2, f_{eq4}^1 + f_{eq2}^2, R_{y3} + f_{eq3}^2, M_{ez3} + f_{eq4}^2, R_{x4}, R_{y4} \rfloor^T$$

Boundary conditions:

$$D_1 = D_2 = D_3 = D_5 = D_6 = D_7, = D_8 = D_9 = 0$$

	2400	9600		-2400	0				
	9600	38400		-9600	0				
			0	0				0	0
	-2400	-9600		2400	0				
				2400	0	-2400	9600		
$[K] =$			0	10000				0	-10000
	0	0		0	0				
				0	0	0	0		
				-2400	0	2400	-9600		
				9600	0	-9600	38400		
			0	0				0	0
			0	-10000				0	10000

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \begin{bmatrix} 2400 & 9600 & 0000 & -2400 & 0000 & 0000 & 0000 & 0000 & 0000 \\ 9600 & 38400 & 0 & -9600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2400 & -9600 & 0 & 14800 & 0 & -2400 & 9600 & 0 & -10000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2400 & 0 & 2400 & -9600 & 0 & 0 \\ 0 & 0 & 0 & 9600 & 0 & -9600 & 34800 & 0 & 0 \\ 0000 & 0000 & 0000 & 0000 & 0000 & 00000 & 0000 & 0000 & 0000 \\ 0 & 0 & 0 & -10000 & 0 & 0 & 0 & 0 & 10000 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix}$$

$$\{2.5\} = [14800]\{D_4\}$$

$$D_4 = v_2 = \frac{2.5}{14800} = 1.689 \times 10^{-4} m$$

For the element 1:

$$d_4 = \theta_2 = -\frac{3}{2L}d_1 - \frac{1}{2}d_2 + \frac{3}{2L}d_3 = \frac{3}{8} \times 1.689 \times 10^{-4} = 0.633375 \times 10^{-4} rd$$

$$\begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^1 \begin{bmatrix} 2400 & 9600 & -2400 & 0 \\ 9600 & 38400 & -9600 & 0 \\ -2400 & -9600 & 2400 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.0001689 \\ 0.0000633375 \end{Bmatrix} - \begin{Bmatrix} -6.875 \\ -7.5 \\ -3.125 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6.47 \\ 5.879 \\ 3.530 \\ 0 \end{Bmatrix}$$

For the element 2:

$$d_2 = \theta_1 = -\frac{3}{2L}d_1 + \frac{3}{2L}d_3 - \frac{1}{2}d_4 = -\frac{3}{8} \times 1.689 \times 10^{-4} = -0.633375 \times 10^{-4} rd$$

$$\begin{Bmatrix} T_{y1} \\ M_{z1} \\ T_{y2} \\ M_{z2} \end{Bmatrix}^2 \left[\begin{bmatrix} 2400 & 0 & -2400 & 9600 \\ 0 & 0 & 0 & 0 \\ -2400 & 0 & 2400 & -9600 \\ 9600 & 0 & -9600 & 38400 \end{bmatrix}\right] \begin{Bmatrix} 0.0001689 \\ -0.0000633375 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 5.625 \\ 0 \\ -5.625 \\ 2.5 \end{Bmatrix}$$

$$= \begin{Bmatrix} -5.220 \\ 0 \\ 5.220 \\ -0.880 \end{Bmatrix}$$

General bending bar element (Timoshenko element):

the displacement field is:

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \ \ et\ \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{v}(\boldsymbol{x})$$

$$\frac{-u(x,y)}{GP} \approx \frac{-u(x,y)}{y} = tg(\theta_z) = \theta_z(x)$$

$$\boldsymbol{u}(\boldsymbol{x}) = -\boldsymbol{y}\boldsymbol{\theta}_{\boldsymbol{z}}(\boldsymbol{x})$$

the deformation field is:

$$\boldsymbol{\varepsilon}_{xx} = \boldsymbol{\varepsilon}_{x} = \frac{\partial u}{\partial x} = -\boldsymbol{y}\frac{\partial \theta_z(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

$$2\boldsymbol{\varepsilon}_{xy} = \boldsymbol{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\boldsymbol{\theta}_{z}(\boldsymbol{x}) + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}$$

the stress field is:

$$\boldsymbol{\sigma}_{xx} = \boldsymbol{\sigma}_{x} = E\varepsilon_x = \boldsymbol{E}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = -\boldsymbol{y}\boldsymbol{E}\frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{x}}$$

$$\boldsymbol{\sigma}_{xy} = \boldsymbol{\tau}_{xy} = Gk_y\gamma_{xy} = \boldsymbol{G}\boldsymbol{k}_{y}\left(-\boldsymbol{\theta}_{z}(\boldsymbol{x}) + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)$$

The internal forces are:

$$\boldsymbol{N} = \int_A \boldsymbol{\sigma}_x \boldsymbol{dA} = -\frac{\partial \boldsymbol{\theta}_z(\boldsymbol{x})}{\partial \boldsymbol{x}}\int_A \boldsymbol{y}\boldsymbol{dA} = \boldsymbol{0}$$

$$\boldsymbol{T}_y = \int_A \boldsymbol{\tau}_{xy}\boldsymbol{dA} = \boldsymbol{GA}\boldsymbol{k}_y\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} - \boldsymbol{\theta}_z(\boldsymbol{x})\right)$$

$$\boldsymbol{M}_z = \boldsymbol{EI}_z\frac{\partial \boldsymbol{\theta}_z(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

The equilibrium equations become for an unloaded element:

$$\frac{\boldsymbol{dT}_y}{\boldsymbol{dx}} = \boldsymbol{0} \rightarrow \frac{\partial}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} - \boldsymbol{\theta}_z(\boldsymbol{x})\right) = \boldsymbol{0} \rightarrow \boldsymbol{\theta}_z(\boldsymbol{x}) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \boldsymbol{c} \rightarrow$$

$$\boldsymbol{T}_y = -\boldsymbol{cGA}\boldsymbol{k}_y$$

$$\boldsymbol{T}_y = \frac{-\boldsymbol{dM}_z}{\boldsymbol{dx}} \rightarrow -\boldsymbol{k}_y = -\boldsymbol{EI}_z\frac{\partial^2 \boldsymbol{\theta}_z(\boldsymbol{x})}{\partial \boldsymbol{x}^2} \rightarrow$$

$$\frac{\partial^2 \boldsymbol{\theta}_z(\boldsymbol{x})}{\partial \boldsymbol{x}^2} = \frac{\boldsymbol{cGA}\boldsymbol{k}_y}{\boldsymbol{EI}_z}$$

$$\boldsymbol{v}(\boldsymbol{x}) = a_0 + a_1x + a_2x^2 + a_3x^3 = \lfloor \boldsymbol{1} \quad \boldsymbol{x} \quad \boldsymbol{x}^2 \quad \boldsymbol{x}^3 \rfloor \{\boldsymbol{a}\}$$

$$\boldsymbol{\theta}_z(\boldsymbol{x}) = \frac{dv}{dx} + c = a_1 + c + 2a_2x + 3a_3x^2$$

$$\boldsymbol{\theta}_z''(\boldsymbol{x}) = 6a_3 = \frac{cGAk_y}{EI_z} \rightarrow \boldsymbol{c} = \frac{\boldsymbol{6EI}_z}{\boldsymbol{GA}\boldsymbol{k}_y}a_3 = \frac{\boldsymbol{L}^2\emptyset}{\boldsymbol{2}}\boldsymbol{a}_3 \ noting\ \frac{\boldsymbol{12EI}_z}{\boldsymbol{L}^2\boldsymbol{GA}\boldsymbol{k}_y} : \emptyset$$

$$\theta_z(x) = \frac{dv}{dx} + c = a_1 + 2a_2x + a_3(3x^2 + \frac{L^2\emptyset}{2})$$

$$\theta_z(x) = \left\lfloor \boldsymbol{0} \quad \boldsymbol{1} \quad \boldsymbol{2x} \quad (\boldsymbol{3x}^2 + \frac{\boldsymbol{L}^2\emptyset}{\boldsymbol{2}}) \right\rfloor \{\boldsymbol{a}\}$$

Boundary Conditions:

Node 1: $x = 0\,,\ v(0) = v_1$ *and* $\theta(0) = \theta_1$;

Node 2 : $x = L\,,\ v(L) = v_2$ *and* $\theta(L) = \theta_2$

These four conditions and the equation $\boldsymbol{c} = \frac{L^2\emptyset}{2} a_3$ *allow to determine the* $\boldsymbol{a_i}$ *and consequently* $\boldsymbol{N_i}$

$$[N_{v1}] = \frac{1}{(1+\emptyset)}\left(1+\emptyset-\frac{\emptyset x}{L}-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}\right),$$

$$[N_{v2}] = \frac{1}{(1+\emptyset)}\left(\frac{(2+\emptyset)x}{2}-\frac{(4+\emptyset)x^2}{2L}+\frac{x^3}{L^2}\right)$$

$$[N_{v3}] = \frac{1}{(1+\emptyset)}\left(\frac{\emptyset x}{L}+\frac{3x^2}{L^2}-\frac{2x^3}{L^3}\right)$$

$$[N_{v4}] = \frac{1}{(1+\emptyset)}\left(-\frac{\emptyset x}{2}-\frac{(2-\emptyset)x^2}{2L}+\frac{x^3}{L^2}\right)$$

$$[N_{\theta 1}] = \frac{1}{(1+\emptyset)}\left(-\frac{6x}{L^2}+\frac{6x^2}{L^3}\right)$$

$$[N_{\theta 2}] = \frac{1}{(1+\emptyset)}\left(1+\emptyset-\frac{(4+\emptyset)x}{L}+\frac{3x^2}{L^2}\right)$$

$$[N_{\theta 3}] = \frac{1}{(1+\emptyset)}\left(\frac{6x}{L^2}-\frac{6x^2}{L^3}\right)$$

$$[N_{\theta 4}] = \frac{1}{(1+\emptyset)}\left(-\frac{(2-\emptyset)x}{L}+\frac{3x^2}{L^2}\right)$$

$$\varepsilon_{xx} = \varepsilon_x = \frac{\partial u}{\partial x} = -y\frac{\partial\theta_z(x)}{\partial x}$$

$$2\varepsilon_{xy} = \gamma_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = Gk_y\left(-\theta_z(x)+\frac{\partial v}{\partial x}\right)$$

$$\begin{Bmatrix}\varepsilon_{xx}\\ 2\varepsilon_{xy}\end{Bmatrix} = [B]\{d\} = \begin{bmatrix}-yN'_{\theta 1} & -yN'_{\theta 2} & -yN'_{\theta 3} & -yN'_{\theta 4}\\ -N_{\theta 1}+N'_{v1} & -N_{\theta 2}+N'_{v2} & -N_{\theta 3}+N'_{v3} & -N_{\theta 4}+N'_{v4}\end{bmatrix}\{d\}$$

$$[B] = \begin{bmatrix}-yN'_{\theta 1} & -yN'_{\theta 2} & -yN'_{\theta 3} & -yN'_{\theta 4}\\ -N_{\theta 1}+N'_{v1} & -N_{\theta 2}+N'_{v2} & -N_{\theta 3}+N'_{v3} & -N_{\theta 4}+N'_{v4}\end{bmatrix}$$

$$[C] = \begin{Bmatrix}E\\ Gk_y)\end{Bmatrix}$$

$$[k] = \frac{EI_z}{L^3(1+\emptyset)} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+\emptyset) & -6L & L^2(2-\emptyset) \\ -12 & -6L & 12 & -6L \\ 6L & (2-\emptyset)L^2 & -6L & L^2(4+\emptyset) \end{bmatrix};$$

$$\emptyset = \frac{12EI_z}{L^2GAk_y}$$