Appendix B: Approximate methods

1.1 Introduction:

Practical problems in Civil Engineering, as in many other domains, are often described by differential or partial differential equations, with boundary conditions and/or initial conditions.

Examples:

Deflection of a bending beam: $EI\frac{d^4v(x)}{dx^4} + q(x) = 0,$

Deflection of a bending plate: $D\nabla^4 w(x,y) + p(x,y) = 0$

One-way consolidation of soil: $c_v \frac{\partial^2 u(z,t)}{\partial z^2} - \frac{\partial u(z,t)}{\partial t} = 0$

For a small number of simple problems, it is possible to obtain an exact solution by directly integrating the differential or partial differential equations that governs them.

However, most practical problems are complex due to the geometry, loads and/or boundary conditions, making direct integration difficult or even impossible. For this reason, several approximate methods have been proposed, including the weighted residuals method, various variational methods, the finite difference method, and the finite element method, among others.

In the following we will describe some of these methods: those that are closely related to the finite element method: the weighted residual method and the Ritz variational method

1.2 Weighted Residual method

For a problem governed by a differential equation or partial differential equation:

$$\boldsymbol{D}(U) = 0, \textit{ on the domain } \boldsymbol{\Omega}$$

With boundary conditions:

$$\boldsymbol{C}(\boldsymbol{U},\boldsymbol{F}) = \boldsymbol{0}, \textit{ on his boundary } \boldsymbol{\Gamma}$$

Where $\boldsymbol{D}$ *and* $\boldsymbol{C}$ *are differential operators.*

If we choose an approximate function $\overline{\boldsymbol{U}}$ *of* $\boldsymbol{U}$*, with unknown parameters* $\boldsymbol{a_i}$*, after substitution in the differential equation of the problem we will have a non-zero residue* $\boldsymbol{R}$ *since* $\overline{U}$ *is not an exact solution*

$$\boldsymbol{R} = \boldsymbol{D}(\overline{\boldsymbol{U}}) \neq \boldsymbol{0}$$

In order to approach the exact solutions, we will determine the parameters $\boldsymbol{a_i}$ *in such a way as to minimize the integral of the residuals over the relevant domain.*

$$\int_{\boldsymbol{\Omega}} \boldsymbol{R d\Omega} = \boldsymbol{minimum}$$

The chances of this rapprochement can be increased by associating a weighting function $\boldsymbol{\phi}$ *with the residual function* $\boldsymbol{R}$ *and minimizing their product; these chances increase further if this new function is chosen in such a way as to obtain a zero minimum.*

$$\int_{\boldsymbol{\Omega}} \boldsymbol{\phi R d\Omega} = \boldsymbol{0}$$

This last equation is the strong form of the weighted residual method and concerns only the interior points of the domain without reference to boundary conditions (such as for example the forces applied to the ends of a bending beam).

To introduce them, we integrate the above equation by parts until they appear. We will therefore have the weak form which gives integral expressions for the domain and for its boundaries.

The choice of the weighting function can be made in different ways, each of which provides a variation of the weighted residual methods.

1.2.1 Collocation Method

The delta ***Dirac function (****Mathematically speaking it is the Dirac distribution) is used as a weighting function: to illustrate the collocation method let us consider the case of a one-dimensional problem defined on a domain* $[\boldsymbol{a}\,, \boldsymbol{b}]$*, the Dirac delta function is written as* $\boldsymbol{\delta(x - x_i)}$*), and the equation of the weighted residuals in its strong form will be*

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{\delta(x - x_i) R(x) dx} = \boldsymbol{0}$$

The Dirac function, considered as the continuous version of the discrete Kronecker delta δ_{ij}, admits similar properties to this one, notably as the Kronecker delta kills the sum:

$$\boldsymbol{\sum_{n=m}^{N} C_i\delta_{ij} = \begin{Bmatrix} \begin{cases} C_j & si\ j \in D, D = m, m+1, m+2, \dots\dots.., N-1, N \\ 0 & si\ j\ \notin D \end{cases} \end{Bmatrix}}$$

*The Dirac **delta kills the integral**:*

$$\boldsymbol{\int_a^b \delta(x - x_i) f(x) dx = \begin{Bmatrix} \begin{cases} f(x_i) & si\ x_i \in [a, b] \\ 0 & si\ x_i \notin [a, b] \end{cases} \end{Bmatrix}}$$

According to this property:

$$\boldsymbol{\int_a^b \delta(x - x_i) R(x) dx = R(x_i)}$$

So

$$\boldsymbol{R(x_i) = 0}$$

The collocation method therefore, consists of choosing, a number of points $\boldsymbol{x_i}$ belonging to the domain of the problem (collocation points), equal to the number of unknown parameters $\boldsymbol{a_i}$ appearing in the expression of the approximate function, so as to form a system of equations allowing the determination of the parameters $\boldsymbol{a_i}$.

1.2.2 Least Squares Method

The weighting function is determined from the residual:

$$\emptyset_i = \frac{dR}{da_i},$$

The equation of the weighted residuals in its strong form will be:

$$\int_a^b \frac{dR}{da_i} R(x) dx = 0$$

$$\boldsymbol{\frac{d}{da_i}\int_a^b R^2(x) dx = 0}$$

Which amounts to minimizing the sum of the squares of the residues (errors), we obtain a system of equations allowing the determination of the parameters $\boldsymbol{a_i}$**.**

1.2.3 Galerkin Method:

In this method the weighting function is obtained from the approximate function (or test function)

$$\emptyset_{\boldsymbol{i}} = \frac{\boldsymbol{d\bar{U}}}{\boldsymbol{da_i}}$$

The equation of the weighted residuals in its strong form will be:

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \frac{\boldsymbol{d\bar{U}}}{\boldsymbol{da_i}} \boldsymbol{R(x)dx = 0}$$

Example:

Let's consider the following simple case:

A simply supported beam of length $\boldsymbol{L = 6m}$**,** *subjected to a uniformly distributed load,* $\boldsymbol{q = 12kN/m}$*,, we want to determine the expression giving the moment at a point of the beam. The differential equation of the problem is:*

$$\frac{d^2M(x)}{dx^2} = q(x)$$

With the boundary conditions: $M(0) = M(L) = 0$

Solution by directs methods:

Strength of materials gives us the expression for the bending moment acting on a section of abscissa x from the left end of the beam:

$$\boldsymbol{M(x)} = q\frac{x^2}{2} - q\frac{L}{2}x = \boldsymbol{6x^2 - 36x}$$

Direct integration of the differential equation of the problem gives:

$$\boldsymbol{\frac{d^2M}{dx^2} = 12} \quad or \quad \boldsymbol{\frac{d^2M}{dx^2} - 12 = 0}$$

$$\frac{dM(x)}{dx} = 12x + a \quad \rightarrow \quad M(x) = 6x^2 + ax + b$$

$$M(0) = 0 \quad from\ where\ \ b = 0 \quad and\ \ M(6) = 216 + 6a = 0 \quad d'ou\ \ a\ = -36$$

So

$$\boldsymbol{M(x) = 6x^2 - 36x}$$

Solution by approximate methods:

Let's consider the approximate function $\overline{\boldsymbol{M}}$ *satisfying the boundary conditions on M(x)*

$$\boldsymbol{\overline{M} = x(x-6)(a_1 + a_2x) = a_2x^3 + (a_1 - 6a_2)x^2 - 6a_1x}$$

By substitution of $\overline{M}$ *in the differential equation:*

$$\frac{d^2}{dx^2}\boldsymbol{(a_2x^3 + (a_1 - 6a_2)x^2 - 6a_1x) - 12 = 6a_2x + 2(a_1 - 6a_2) - 12 \neq 0}$$

We will have a residue R

$$\boldsymbol{R = 6a_2x + 2(a_1 - 6a_2) - 12}$$

Resolution by the collocation method:

Let's choose $x = 2$ *and* $x = 4$ *as collocation points*

So:

$$R(2) = 2a_1 - 12 = 0$$

$$d'o\text{ù}\ \ \boldsymbol{a_1 = 6}$$

$$R(4) = 2a_1 + 12a_2 - 12 = 0$$

$$d'o\text{ù}\ \ \boldsymbol{a_2 = 0}$$

So $$\boldsymbol{\overline{M}(x) = 6x(x-6) = 6x^2 - 36x}$$

Solving by the least squares method

$$R = 6a_2x + 2(a_1 - 6a_2 - 6)$$

$$R^2 = 36a_2^2x^2 + 24(a_1 - 6a_2 - 6)a_2x + 4(a_1 - 6a_2 - 6)^2$$

$$\int_0^6 R^2\,dx = [12a_2^2x^3 + 12(a_1 - 6a_2 - 6)a_2x^2 + 4(a_1 - 6a_2 - 6)^2x]_0^6$$

$$\int_0^6 R^2\,dx = 2592a_2^2 + 432a_2(a_1 - 6a_2 - 6) + 24(a_1 - 6a_2 - 6)^2$$

$$\frac{d}{da_1}\int_0^6 R^2\,dx = 0 \;\rightarrow 432a_2 + 24(a_1 - 6a_2 - 6) = 0$$

$$24a_1 + 288a_2 = 144 \qquad \rightarrow \qquad \boldsymbol{a_1 + 12a_2 = 6} \ldots\ldots\ldots\ldots\ldots \quad (a)$$

$$\frac{d}{da_2}\int_0^6 R^2\,dx = 0 \;\rightarrow 5184a_2 + 432(a_1 - 6a_2 - 6) - 144(a_1 - 6a_2 - 6) = 0$$

$$288a_1 + 6480a_2 = 1728 \qquad \rightarrow \qquad \boldsymbol{2a_1 + 45a_2 = 12} \ldots\ldots\ldots\ldots \quad (b)$$

From (a) and (b) we will have:

$$\boldsymbol{a_1 = 6 \quad et \quad a_2 = 0}$$

So: $$\boldsymbol{\bar{M}(x) = 6x(x - 6) = 6x^2 - 36x}$$

Resolution by Galerkin method:

$$\boldsymbol{\emptyset_1 = \frac{d\bar{M}}{da_1} = x^2 - 6x} \qquad \boldsymbol{and} \qquad \boldsymbol{\emptyset_2 = \frac{d\bar{M}}{da_2} = x^3 - 6x^2}$$

$$\int_{\mathbf{0}}^{\mathbf{6}} \boldsymbol{\emptyset_1 R dx} = 0 \;\; gives \quad \int_0^6 (x^2 - 6x)(6a_2x + 2(a_1 - 6a_2 - 6))dx$$

$$\int_0^6 (6a_2x^3 + 2(a_1 - 24a_2 - 6)x^2 - 12(a_1 - 6a_2 - 6)x)dx$$

$$[\frac{3}{2}a_2x^4 + \frac{2}{3}(a_1 - 24a_2 - 6)x^3 - 6(a_1 - 6a_2 - 6)x^2]_0^6$$

$$\boldsymbol{a_1 + 3a_2 = 6} \qquad \ldots. \;(c)$$

$$\int_0^6 \emptyset_2 \boldsymbol{Rdx} = 0 \;\; gives \quad \int_0^6 (x^3 - 6x^2)(6a_2x + 2(a_1 - 6a_2 - 6))dx$$

$$[\frac{6}{5}a_2x^5 + \frac{1}{2}(a_1 - 24a_2 - 6)x^4 - 4(a_1 - 6a_2 - 6)x^3]_0^6$$

$$\boldsymbol{a_1 + \frac{72}{5}a_2 = 6} \qquad \dots\dots(d)$$

from (c) and (d) we will have: $\boldsymbol{a_1 = 6}$ $\quad$ ***et*** $\quad$ $\boldsymbol{a_2 = 0}$

So: $\quad\boldsymbol{\bar{M}(x) = 6x(x-6) = 6x^2 - 36x}$

1.3 VARIATIONAL METHODS:

Let's consider a differential equation of the form:

$$\boldsymbol{L\,(u) \;=\; 0}$$

Where **L** *is a differential operator.*

For this differential equation, if a functional π exists / that:

$$\boldsymbol{\pi = \int_a^b f[x, u, u'(x)]dx}$$

Then among the admissible functions $\boldsymbol{u}\ (\boldsymbol{x})$*, the function that gives the most approximate solution to the equation*
$\boldsymbol{L\,(u) \;=\; 0}$ *is the one that gives the integral π in the interval* $[a\,,b]$ *the extreme value (maximum or minimum) for a given functional* $\boldsymbol{f}$*. The function* $\boldsymbol{u(x)}$*, which maximizes or minimizes f, is called the stationary function.*

In variational calculus, it can be shown that the stationary function

$$\boldsymbol{f[x, u, u'(x)]dx}$$

must satisfy the Euler-Lagrange equation

$$\boldsymbol{\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0}$$

In the variational approach, the nodal values of u_i *must be found in such a way as to make the functional* $\pi(u)$ *stationary such that:*

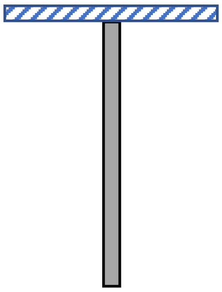
$$\boldsymbol{\delta\pi(u) = \sum_{k=1}^{n} \frac{\partial\pi}{\partial u_k}\delta u_k = 0}$$

where n *is the number of discrete solutions* u_i *assigned to the domain. From the previous equation it follows that*

$$\boldsymbol{\frac{\partial\pi}{\partial u_k} = 0, \quad k = 1, 2, \dots\dots\dots; n}$$

Example:

A suspended bar, of volumetric weight $\boldsymbol{\rho g}$*:*



The differential equation is:

$$EA\frac{d^2u}{dx^2}+\rho gA=0 \;\rightarrow\; \frac{d^2u}{dx^2}=-\frac{\rho g}{E} \;\; with \;\; u(0)=0 \quad and \;\; EA\left.\frac{du}{dx}\right)_h=\left.\frac{du}{dx}\right)_h=0$$

The exact solution of the problem is:

$$\frac{du}{dx}=-\frac{\rho g}{E}x+c_1$$

$$u(x)=-\frac{\rho g}{2E}x^2+c_1x+c_2$$

$$As \;\; u(0)=0 \; then \;\; c_2=0 \; and \; as \; \left.\frac{du}{dx}\right)_h=0 \; then \;\; c_1=\frac{\rho gh}{E}$$

$$\boldsymbol{u(x)=\frac{\rho gh}{E}x-\frac{\rho g}{2E}x^2}$$

Approximate solution

The functional $\boldsymbol{\pi}$ *here is the potential energy of the bar:*

$$\pi(u)=E(u)=\frac{1}{2}\int_0^h A\sigma\varepsilon dx-\int_0^h \rho gAu(x)dx$$

$$\boldsymbol{E(u)=\int_0^h\left[\frac{1}{2}EA\left(\frac{du}{dx}\right)^2-\rho gAu(x)\right]dx}$$

Let's take the quadratic approximate function:

$$\boldsymbol{\bar{u}(x)=a_0+a_1x+a_2x^2}$$

This function must minimize the energy $\boldsymbol{E(u)}$ *and must satisfy the boundary conditions:*

$$\bar{u}(0)=0 \quad \rightarrow a_0=0$$

$$\left.\frac{d\bar{u}}{dx}\right)_h = 0 \quad \rightarrow \quad \boldsymbol{a_1 + 2a_2h = 0} \quad \rightarrow \quad \boldsymbol{a_2 = -a_1/2h}$$

$$E(\bar{u}) = \int_0^h \left[\frac{1}{2}EA\left(\frac{d\bar{u}}{dx}\right)^2 - \rho gA\bar{u}(x)\right]dx$$

$$\boldsymbol{E(\bar{u}) = \int_0^h \left[\frac{1}{2}EA\,(a_1 + 2a_2x)^2 - \rho gA(a_1x + a_2x^2)\right]dx}$$

$$E(\bar{u}) = \frac{1}{2}\left[EA\left({a_1}^2h + 2a_1a_2h^2 + 4{a_2}^2\frac{h^3}{3}\right) - \rho gA(a_1\frac{h^2}{2} + a_2\frac{h^3}{3})\right]$$

$$\frac{\partial E(\bar{u})}{\partial a_1} = \frac{1}{2}EA(2a_1h + 2a_2h^2) - \rho gA\frac{h^2}{2} = 0$$

$$\boldsymbol{a_1 + a_2h = \frac{\rho gh}{2E}}\,, \qquad (1)$$

By replacing in this equation $\boldsymbol{a_2}$ *with* $\boldsymbol{-a_1/2h}$ *we get:*

$$\boldsymbol{a_1 - \left(\frac{a_1}{2h}\right)h = \frac{\rho gh}{2E}} \quad \rightarrow \boldsymbol{a_1 = \frac{\rho gh}{E}}$$

$$\boldsymbol{a_2 = -\frac{a_1}{2h} = -\frac{\rho g}{2E}}$$

$$\rightarrow \qquad \boldsymbol{u(x) = \frac{\rho gh}{E}x - \frac{\rho g}{2E}x^2}$$

Which is the previous exact solution.

Also, we can write a second equation using $\frac{\partial E(\bar{u})}{\partial a_2}$

$$\frac{\partial E(\bar{u})}{\partial a_2} = \frac{1}{2}EA\left(2a_1h^2 + 8a_2\frac{h^3}{3}\right) - \rho gA\frac{h^3}{3} = 0$$

$$a_1 + 4a_2\frac{h}{3} = \frac{\rho gh}{3E}\,, \qquad (2)$$

From (1) and (2) we will have:

$$\boldsymbol{a_1 = \frac{\rho gh}{E}\ \ ,\ \ et\ a_2 = -\frac{\rho g}{2E}}$$

So, the approximate solution is:

$$\boldsymbol{\bar{u}(x) = \frac{\rho g h}{E}x - \frac{\rho g}{2E}x^2}$$

Exercise:

A beam fixed at its left end and free at its right end and supports a constant transverse load q*.Determine an approximate solution of the deflection* w

The differential equation of the bending of beams with constant stiffness EI *is*

$$\boldsymbol{\frac{d^4w}{dx^4} = -\frac{q(x)}{EI}}$$

With boundary conditions:

$$\boldsymbol{w(0) = 0, \varphi(0) = -\left.\frac{dw}{dx}\right)_0 = 0, M(L) = EI\left.\frac{d^2w}{dx^2}\right)_L = 0,}$$

$$\boldsymbol{T(L) = EI\left.\frac{d^3w}{dx^3}\right)_L = 0}$$

$$\boldsymbol{\bar{w}(x) = a_0 + a_1x + a_2x^2 + a_3x^3}$$