MECANIQUE DES MILIEUX CONTINUS

Chapitre 1
Préliminaires mathématiques

Exercices

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Ex. 1 Soient deux points $M(4{,}3{,}0)$ et $N(3{,}4{,}0)$ définis dans un repère orthonormé $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$.

On demande de :

1. Représenter les vecteurs $\bar{V} = \overline{OM}$ et $\bar{W} = \overline{ON}$, puis calculer leurs normes $r_{\bar{V}}$ et $r_{\bar{W}}$.
2. Calculer le produit scalaire $\bar{V} \cdot \bar{W}$.
3. Calculer le produit vectoriel $\bar{U} = \bar{V} \wedge \bar{W}$.
4. Calculer le produit mixte $(\bar{U}, \bar{V}, \bar{W})$.
5. Calculer le produit tensoriel $\bar{\bar{T}} = \bar{V} \otimes \bar{W}$.
6. Calculer la trace du tenseur $\bar{\bar{T}}$.
7. Calculer le déterminant du tenseur $\bar{\bar{T}}$.

Rép. 1

1. Représentation des vecteurs et calcul de leurs normes

- Vecteurs :

  ▻ $\bar{V} = \overline{OM} = \begin{Bmatrix} 4 \\ 3 \\ 0 \end{Bmatrix} = {}^t\{4 \quad 3 \quad 0\} \quad / \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$

  ▻ $\bar{W} = \overline{ON} = \begin{Bmatrix} 3 \\ 4 \\ 0 \end{Bmatrix} = {}^t\{3 \quad 4 \quad 0\} \quad / \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$

- Normes :

  Norme du vecteur $\bar{A}$: $\boldsymbol{r = \|\bar{A}\| = \sqrt{A_p A_p} \quad | \quad p = 1, 2, 3}$

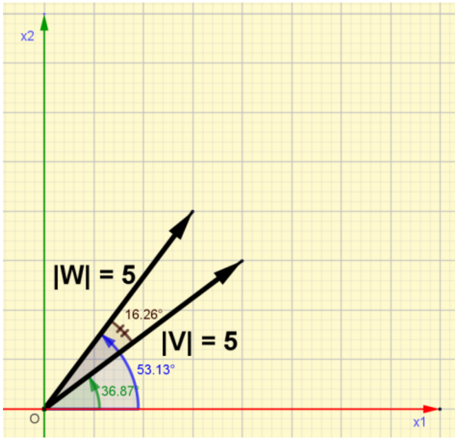
  $\Rightarrow \quad r_{\bar{V}} = \|\bar{V}\| = \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{4^2 + 3^2 + 0^2} = 5$

  $\Rightarrow \quad r_{\bar{W}} = \|\bar{W}\| = \sqrt{W_1^2 + W_2^2 + W_3^2} = \sqrt{3^2 + 4^2 + 0^2} = 5$

- Remarque :

  $r_{\bar{V}} = r_{\bar{W}} = 5$

  $\Rightarrow$ Les deux vecteurs ont la même longueur.



2. Produit scalaire : $\bar{V} \cdot \bar{W}$

$$\boldsymbol{\bar{V} \cdot \bar{W} = V_p W_p} \quad \Rightarrow \quad \bar{V} \cdot \bar{W} = V_1 W_1 + V_2 W_2 + V_3 W_3 = 4 \times 3 + 3 \times 4 + 0 \times 0 = 24$$

Vérification 1 :

$$\bar{V} \cdot \bar{W} = \|\bar{V}\| \times \|\bar{W}\| \times cos(\alpha) \quad | \quad \alpha = \alpha_W - \alpha_V = 53.13° - 36.87° = 16.26°$$

$$\Rightarrow \quad cos(\alpha) = \frac{\bar{V} \cdot \bar{W}}{\|\bar{V}\| \times \|\bar{W}\|} = \frac{24}{5 \times 5} = 0.96 \quad \Rightarrow \quad \alpha = cos^{-1}(0.96) = 16.26 \qquad Ok.$$

Vérification 2 :

$$\boldsymbol{\bar{V} \cdot \bar{W} = \frac{1}{4}(\|\bar{V} + \bar{W}\|^2 - \|\bar{V} - \bar{W}\|^2)} \quad | \quad \bar{V} + \bar{W} = {}^t\{7 \quad 7 \quad 0\} \text{ et } \bar{V} - \bar{W} = {}^t\{1 \quad 1 \quad 0\}$$

$$\Rightarrow \quad \bar{V} \cdot \bar{W} = \frac{1}{4}\left[\left(\sqrt{7 \times 7 + 7 \times 7 + 0 \times 0}\right)^2 - \left(\sqrt{1 \times 1 + 1 \times 1 + 0 \times 0}\right)^2\right]^2 = \frac{1}{4}(98 - 2)^2 = 24 \qquad Ok.$$

3. Produit vectoriel : $\bar{\boldsymbol{U}} = \bar{\boldsymbol{V}} \wedge \bar{\boldsymbol{W}}$

$$\bar{\boldsymbol{U}} = \bar{\boldsymbol{V}} \wedge \bar{\boldsymbol{W}} = \begin{vmatrix} \bar{\boldsymbol{x}}_1 & \bar{\boldsymbol{x}}_2 & \bar{\boldsymbol{x}}_3 \\ \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \\ \boldsymbol{W}_1 & \boldsymbol{W}_2 & \boldsymbol{W}_3 \end{vmatrix} = \begin{Bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \boldsymbol{U}_3 \end{Bmatrix} = {}^t\{\boldsymbol{U}_1 \quad \boldsymbol{U}_2 \quad \boldsymbol{U}_3\} \quad / \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

$$\Rightarrow \quad \bar{U} = \bar{V} \wedge \bar{W} = \begin{vmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ 4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 4 & 0 \end{vmatrix} \bar{x}_1 - \begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix} \bar{x}_2 + \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} \bar{x}_3 = 0\bar{x}_1 + 0\bar{x}_2 + 7\bar{x}_3$$

$$\Rightarrow \quad \bar{U} = \begin{Bmatrix} 0 \\ 0 \\ 7 \end{Bmatrix} = {}^t\{0 \quad 0 \quad 7\} \quad / \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

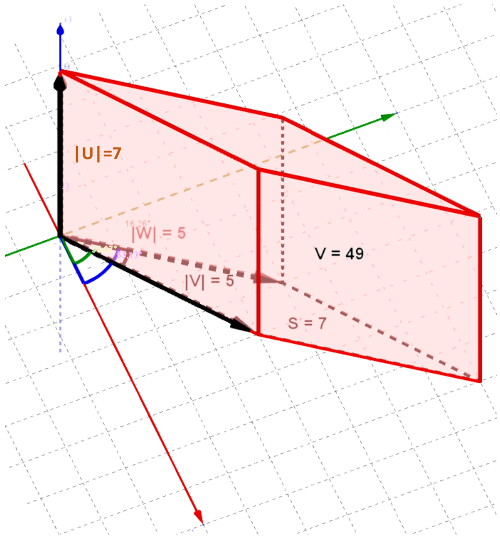
<u>Vérification</u> :

$$\bar{\boldsymbol{U}} = \bar{\boldsymbol{V}} \wedge \bar{\boldsymbol{W}} = \boldsymbol{\epsilon_{ijk} V_j W_k}$$

$$\Rightarrow \quad \begin{cases} U_1 = \epsilon_{123} V_2 W_3 + \epsilon_{132} V_3 W_2 = +1 \times 3 \times 0 - 1 \times 0 \times 4 = 0 \\ U_2 = \epsilon_{231} V_3 W_1 + \epsilon_{213} V_1 W_3 = +1 \times 0 \times 3 - 1 \times 4 \times 0 = 0 \\ U_3 = \epsilon_{312} V_1 W_2 + \epsilon_{321} V_2 W_1 = +1 \times 4 \times 4 - 1 \times 3 \times 3 = 7 \end{cases} \quad \mathit{Ok.}$$

<u>Symbole de permutation</u>
$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$
$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$

4. Produit mixte : $(\bar{\boldsymbol{U}}, \bar{\boldsymbol{V}}, \bar{\boldsymbol{W}})$



$$(\bar{\boldsymbol{U}}, \bar{\boldsymbol{V}}, \bar{\boldsymbol{W}}) = \begin{vmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 & \boldsymbol{U}_3 \\ \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \\ \boldsymbol{W}_1 & \boldsymbol{W}_2 & \boldsymbol{W}_3 \end{vmatrix} \quad \Rightarrow \quad (\bar{U}, \bar{V}, \bar{W}) = \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 7 \end{vmatrix}$$

$$\Rightarrow \quad (\bar{U}, \bar{V}, \bar{W}) = \begin{vmatrix} 4 & 0 \\ 0 & 7 \end{vmatrix} \times 4 - \begin{vmatrix} 3 & 0 \\ 0 & 7 \end{vmatrix} \times 3 + \begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix} \times 0$$

$$= 4 \times 7 \times 4 - 3 \times 7 \times 3 + 0 = 49$$

<u>Vérification</u> : $V = |(\bar{U}, \bar{V}, \bar{W})| = S \times h \quad | \quad h = U_3 = 7$

$$S = \|\bar{V}\| \times \|\bar{W}\| \times sin(\alpha) = 5 \times 5 \times sin(16.26) = 7$$

$$\Rightarrow \quad V = 7 \times 7 = 49 \qquad \mathit{Ok.}$$

5. Produit tensoriel : $\bar{\bar{\boldsymbol{T}}} = \bar{\boldsymbol{V}} \otimes \bar{\boldsymbol{W}}$

$$\bar{\bar{\boldsymbol{T}}} = \bar{\boldsymbol{V}} \otimes \bar{\boldsymbol{W}} \quad \Rightarrow \quad \boldsymbol{t_{ij} = V_i W_j} \quad \Rightarrow \quad \bar{\bar{T}} = \begin{bmatrix} 4 \times 3 & 4 \times 4 & 4 \times 0 \\ 3 \times 3 & 3 \times 4 & 3 \times 0 \\ 0 \times 3 & 0 \times 4 & 0 \times 0 \end{bmatrix} = \begin{bmatrix} 12 & 16 & 0 \\ 9 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Trace du tenseur $\bar{\bar{\boldsymbol{T}}}$

$$\boldsymbol{Tr}(\bar{\bar{\boldsymbol{T}}}) = \boldsymbol{t_{ij}} \quad \Rightarrow \quad \boldsymbol{t_{ii} = t_{11} + t_{22} + t_{33}} \quad \Rightarrow \quad t_{ii} = 12 + 12 + 0 = 24$$

<u>Remarque</u> : $\boldsymbol{Tr}(\bar{\boldsymbol{V}} \otimes \bar{\boldsymbol{W}}) = \bar{\boldsymbol{V}} \cdot \bar{\boldsymbol{W}}$

7. Déterminant du tenseur $\bar{\bar{\boldsymbol{T}}}$

$$\boldsymbol{det}(\bar{\bar{\boldsymbol{T}}}) = \begin{vmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} & \boldsymbol{t}_{13} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} & \boldsymbol{t}_{23} \\ \boldsymbol{t}_{31} & \boldsymbol{t}_{32} & \boldsymbol{t}_{33} \end{vmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \begin{vmatrix} 12 & 16 & 0 \\ 9 & 12 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 12 & 0 \\ 0 & 0 \end{vmatrix} \times 12 - \begin{vmatrix} 9 & 0 \\ 0 & 0 \end{vmatrix} \times 16 + \begin{vmatrix} 9 & 12 \\ 0 & 0 \end{vmatrix} \times 0 = 0$$

Vérification :

$$\boldsymbol{det(\bar{\bar{T}}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}t_{ip}t_{jq}t_{kr}}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} \epsilon_{123}\epsilon_{123}t_{11}t_{22}t_{33} + \epsilon_{123}\epsilon_{231}t_{12}t_{23}t_{31} + \epsilon_{123}\epsilon_{312}t_{13}t_{21}t_{32} \\ + \\ \epsilon_{123}\epsilon_{132}t_{11}t_{23}t_{32} + \epsilon_{123}\epsilon_{213}t_{12}t_{21}t_{33} + \epsilon_{123}\epsilon_{321}t_{13}t_{22}t_{31} \\ + \\ \epsilon_{231}\epsilon_{123}t_{21}t_{32}t_{13} + \epsilon_{231}\epsilon_{231}t_{22}t_{33}t_{11} + \epsilon_{231}\epsilon_{312}t_{23}t_{31}t_{12} \\ + \\ \epsilon_{231}\epsilon_{132}t_{21}t_{33}t_{12} + \epsilon_{231}\epsilon_{213}t_{22}t_{31}t_{13} + \epsilon_{231}\epsilon_{321}t_{23}t_{32}t_{11} \\ + \\ \epsilon_{312}\epsilon_{123}t_{31}t_{12}t_{23} + \epsilon_{312}\epsilon_{231}t_{32}t_{13}t_{21} + \epsilon_{312}\epsilon_{312}t_{33}t_{11}t_{22} \\ + \\ \epsilon_{312}\epsilon_{132}t_{31}t_{13}t_{22} + \epsilon_{312}\epsilon_{213}t_{32}t_{11}t_{23} + \epsilon_{312}\epsilon_{321}t_{33}t_{12}t_{21} \\ + \\ \epsilon_{132}\epsilon_{123}t_{11}t_{32}t_{23} + \epsilon_{132}\epsilon_{231}t_{12}t_{33}t_{21} + \epsilon_{132}\epsilon_{312}t_{13}t_{31}t_{22} \\ + \\ \epsilon_{132}\epsilon_{132}t_{11}t_{33}t_{22} + \epsilon_{132}\epsilon_{213}t_{12}t_{31}t_{23} + \epsilon_{132}\epsilon_{321}t_{13}t_{32}t_{21} \\ + \\ \epsilon_{213}\epsilon_{123}t_{21}t_{12}t_{33} + \epsilon_{213}\epsilon_{231}t_{22}t_{13}t_{31} + \epsilon_{213}\epsilon_{312}t_{23}t_{11}t_{32} \\ + \\ \epsilon_{213}\epsilon_{132}t_{21}t_{13}t_{32} + \epsilon_{213}\epsilon_{213}t_{22}t_{11}t_{33} + \epsilon_{213}\epsilon_{321}t_{23}t_{12}t_{31} \\ + \\ \epsilon_{321}\epsilon_{123}t_{31}t_{22}t_{13} + \epsilon_{321}\epsilon_{231}t_{32}t_{23}t_{11} + \epsilon_{321}\epsilon_{312}t_{33}t_{21}t_{12} \\ + \\ \epsilon_{321}\epsilon_{132}t_{31}t_{23}t_{12} + \epsilon_{321}\epsilon_{213}t_{32}t_{21}t_{13} + \epsilon_{321}\epsilon_{321}t_{33}t_{22}t_{11} \end{pmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} (+1)\times(+1)\times 12\times 12\times 0 + (+1)\times(+1)\times 16\times 0\times 0 + (+1)\times(+1)\times 0\times 9\times 0 \\ + \\ (+1)\times(-1)\times 12\times 0\times 0 + (+1)\times(-1)\times 16\times 9\times 0 + (+1)\times(-1)\times 0\times 12\times 0 \\ + \\ (+1)\times(+1)\times 9\times 0\times 0 + (+1)\times(+1)\times 12\times 0\times 12 + (+1)\times(+1)\times 0\times 0\times 16 \\ + \\ (+1)\times(-1)\times 9\times 0\times 16 + (+1)\times(-1)\times 12\times 0\times 0 + (+1)\times(-1)\times 0\times 0\times 12 \\ + \\ (+1)\times(+1)\times 0\times 16\times 0 + (+1)\times(+1)\times 0\times 0\times 9 + (+1)\times(+1)\times 0\times 12\times 12 \\ + \\ (+1)\times(-1)\times 0\times 0\times 12 + (+1)\times(-1)\times 0\times 12\times 0 + (+1)\times(-1)\times 0\times 16\times 9 \\ + \\ (-1)\times(+1)\times 12\times 0\times 0 + (-1)\times(+1)\times 16\times 0\times 9 + (-1)\times(+1)\times 0\times 0\times 12 \\ + \\ (-1)\times(-1)\times 12\times 0\times 12 + (-1)\times(-1)\times 16\times 0\times 0 + (-1)\times(-1)\times 0\times 0\times 9 \\ + \\ (-1)\times(+1)\times 9\times 16\times 0 + (-1)\times(+1)\times 12\times 0\times 0 + (-1)\times(+1)\times 0\times 12\times 0 \\ + \\ (-1)\times(+1)\times 9\times 16\times 0 + (-1)\times(+1)\times 12\times 1\times 0 + (-1)\times(+1)\times 0\times 12\times 0 \\ + \\ (-1)\times(+1)\times 0\times 12\times 0 + (-1)\times(-1)\times 0\times 0\times 12 + (-1)\times(-1)\times 0\times 9\times 16 \\ + \\ (-1)\times(-1)\times 0\times 0\times 16 + (-1)\times(-1)\times 0\times 9\times 0 + (-1)\times(-1)\times 0\times 12\times 12 \end{pmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = 0 \qquad \textit{Ok.}$$

Ex. 2 Soit le tenseur suivant :

$$\bar{\bar{T}} = \begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

On demande de calculer :

1. Sa partie symétrique $\bar{\bar{T}}_s$.
2. Sa partie antisymétrique $\bar{\bar{T}}_a$.
3. Sa trace.
4. Son déterminant.

<u>Rép</u>. 2

1. Partie symétrique $\bar{\bar{T}}_s$ du tenseur $\bar{\bar{T}}$

$$\bar{\bar{\boldsymbol{T}}}_s = \frac{1}{2}\left(\bar{\bar{\boldsymbol{T}}} + {}^t\bar{\bar{\boldsymbol{T}}}\right) \quad \Rightarrow \quad (\boldsymbol{t}_s)_{ij} = \frac{1}{2}\left(\boldsymbol{t}_{ij} + \boldsymbol{t}_{ji}\right)$$

$$\Rightarrow \quad \bar{\bar{T}}_s = \frac{1}{2}\left[\begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right] = \begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \Rightarrow \quad \bar{\bar{T}}_s \equiv \bar{\bar{T}}$$

2. Partie antisymétrique $\bar{\bar{T}}_a$ du tenseur $\bar{\bar{T}}$

$$\bar{\bar{\boldsymbol{T}}}_a = \frac{1}{2}\left(\bar{\bar{\boldsymbol{T}}} - {}^t\bar{\bar{\boldsymbol{T}}}\right) \quad \Rightarrow \quad (\boldsymbol{t}_a)_{ij} = \frac{1}{2}\left(\boldsymbol{t}_{ij} - \boldsymbol{t}_{ji}\right)$$

$$\Rightarrow \quad \bar{\bar{T}}_a = \frac{1}{2}\left[\begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \bar{\bar{T}}_a \equiv \bar{\bar{0}}$$

3. Trace du tenseur $\bar{\bar{T}}$

$$\boldsymbol{Tr}\left(\bar{\bar{\boldsymbol{T}}}\right) = \boldsymbol{t}_{ii} \quad \Rightarrow \quad Tr\left(\bar{\bar{T}}\right) = t_{11} + t_{22} + t_{33} = 8 + 0 + 4 = 12$$

<u>Remarque</u> : $Tr\left(\bar{\bar{T}}\right) = Tr\left(\bar{\bar{T}}_s\right) = 12 \quad \& \quad Tr\left(\bar{\bar{T}}_a\right) = 0$

$\Rightarrow \quad \bar{\bar{T}}$ – Tenseur symétrique ($t_{ij} = t_{ji}$ pour $i \neq j$).

4. Déterminant du tenseur $\bar{\bar{T}}$

$$\boldsymbol{det}\left(\bar{\bar{\boldsymbol{T}}}\right) = \begin{vmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} & \boldsymbol{t}_{13} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} & \boldsymbol{t}_{23} \\ \boldsymbol{t}_{31} & \boldsymbol{t}_{32} & \boldsymbol{t}_{33} \end{vmatrix}$$

$$\Rightarrow \quad det\left(\bar{\bar{T}}\right) = \begin{vmatrix} 8 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 4 \end{vmatrix} \times 8 - \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} \times 3 + \begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} \times 0 = -36$$

<u>Vérification</u> :

$$\boldsymbol{det}\left(\bar{\bar{\boldsymbol{T}}}\right) = \frac{1}{6}\,\boldsymbol{\epsilon}_{ijk}\boldsymbol{\epsilon}_{pqr}\boldsymbol{t}_{ip}\boldsymbol{t}_{jq}\boldsymbol{t}_{kr}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} \epsilon_{123}\epsilon_{123}t_{11}t_{22}t_{33} + \epsilon_{123}\epsilon_{231}t_{12}t_{23}t_{31} + \epsilon_{123}\epsilon_{312}t_{13}t_{21}t_{32} \\ + \\ \epsilon_{123}\epsilon_{132}t_{11}t_{23}t_{32} + \epsilon_{123}\epsilon_{213}t_{12}t_{21}t_{33} + \epsilon_{123}\epsilon_{321}t_{13}t_{22}t_{31} \\ + \\ \epsilon_{231}\epsilon_{123}t_{21}t_{32}t_{13} + \epsilon_{231}\epsilon_{231}t_{22}t_{33}t_{11} + \epsilon_{231}\epsilon_{312}t_{23}t_{31}t_{12} \\ + \\ \epsilon_{231}\epsilon_{132}t_{21}t_{33}t_{12} + \epsilon_{231}\epsilon_{213}t_{22}t_{31}t_{13} + \epsilon_{231}\epsilon_{321}t_{23}t_{32}t_{11} \\ + \\ \epsilon_{312}\epsilon_{123}t_{31}t_{12}t_{23} + \epsilon_{312}\epsilon_{231}t_{32}t_{13}t_{21} + \epsilon_{312}\epsilon_{312}t_{33}t_{11}t_{22} \\ + \\ \epsilon_{312}\epsilon_{132}t_{31}t_{13}t_{22} + \epsilon_{312}\epsilon_{213}t_{32}t_{11}t_{23} + \epsilon_{312}\epsilon_{321}t_{33}t_{12}t_{21} \\ + \\ \epsilon_{132}\epsilon_{123}t_{11}t_{32}t_{23} + \epsilon_{132}\epsilon_{231}t_{12}t_{33}t_{21} + \epsilon_{132}\epsilon_{312}t_{13}t_{31}t_{22} \\ + \\ \epsilon_{132}\epsilon_{132}t_{11}t_{33}t_{22} + \epsilon_{132}\epsilon_{213}t_{12}t_{31}t_{23} + \epsilon_{132}\epsilon_{321}t_{13}t_{32}t_{21} \\ + \\ \epsilon_{213}\epsilon_{123}t_{21}t_{12}t_{33} + \epsilon_{213}\epsilon_{231}t_{22}t_{13}t_{31} + \epsilon_{213}\epsilon_{312}t_{23}t_{11}t_{32} \\ + \\ \epsilon_{213}\epsilon_{132}t_{21}t_{13}t_{32} + \epsilon_{213}\epsilon_{213}t_{22}t_{11}t_{33} + \epsilon_{213}\epsilon_{321}t_{23}t_{12}t_{31} \\ + \\ \epsilon_{321}\epsilon_{123}t_{31}t_{22}t_{13} + \epsilon_{321}\epsilon_{231}t_{32}t_{23}t_{11} + \epsilon_{321}\epsilon_{312}t_{33}t_{21}t_{12} \\ + \\ \epsilon_{321}\epsilon_{132}t_{31}t_{23}t_{12} + \epsilon_{321}\epsilon_{213}t_{32}t_{21}t_{13} + \epsilon_{321}\epsilon_{321}t_{33}t_{22}t_{11} \end{pmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} (+1)\times(+1)\times 8\times 0\times 4 + (+1)\times(+1)\times 3\times 0\times 0 + (+1)\times(+1)\times 0\times 3\times 0 \\ + \\ (+1)\times(-1)\times 8\times 0\times 0 + {\color{red}(+1)\times(-1)\times 3\times 3\times 4} + (+1)\times(-1)\times 0\times 0\times 0 \\ + \\ (+1)\times(+1)\times 3\times 0\times 0 + (+1)\times(+1)\times 0\times 4\times 8 + (+1)\times(+1)\times 0\times 0\times 3 \\ + \\ {\color{red}(+1)\times(-1)\times 3\times 4\times 3} + (+1)\times(-1)\times 0\times 0\times 0 + (+1)\times(-1)\times 0\times 0\times 8 \\ + \\ (+1)\times(+1)\times 0\times 3\times 0 + (+1)\times(+1)\times 0\times 0\times 3 + (+1)\times(+1)\times 0\times 8\times 0 \\ + \\ (+1)\times(-1)\times 0\times 0\times 0 + (+1)\times(-1)\times 0\times 8\times 0 + {\color{red}(+1)\times(-1)\times 4\times 3\times 3} \\ + \\ (-1)\times(+1)\times 8\times 0\times 0 + {\color{red}(-1)\times(+1)\times 3\times 4\times 3} + (-1)\times(+1)\times 0\times 0\times 0 \\ + \\ (-1)\times(-1)\times 8\times 0\times 0 + (-1)\times(-1)\times 3\times 0\times 0 + (-1)\times(-1)\times 0\times 0\times 3 \\ + \\ {\color{red}(-1)\times(+1)\times 3\times 3\times 4} + (-1)\times(+1)\times 0\times 0\times 0 + (-1)\times(+1)\times 0\times 8\times 0 \\ + \\ (-1)\times(+1)\times 3\times 0\times 0 + (-1)\times(+1)\times 0\times 8\times 0 + (-1)\times(+1)\times 0\times 3\times 0 \\ + \\ (-1)\times(+1)\times 0\times 0\times 0 + (-1)\times(-1)\times 0\times 0\times 8 + {\color{red}(-1)\times(-1)\times 4\times 3\times 3} \\ + \\ (-1)\times(-1)\times 0\times 0\times 3 + (-1)\times(-1)\times 0\times 3\times 0 + (-1)\times(-1)\times 0\times 0\times 8 \end{pmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} (+1)\times(-1)\times 3\times 3\times 4 + (+1)\times(-1)\times 3\times 4\times 3 + (+1)\times(-1)\times 4\times 3\times 3 \\ + \\ (-1)\times(+1)\times 3\times 4\times 3 + (-1)\times(+1)\times 3\times 3\times 4 + (-1)\times(+1)\times 4\times 3\times 3 \end{pmatrix}$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}(-36 - 36 - 36 - 36 - 36 - 36)$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = -36 \qquad Ok.$$

Ex. 3 Soit le tenseur suivant :

$$\bar{\bar{T}} = \begin{pmatrix} 8 & 3 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

On demande de calculer :

1. Sa partie symétrique $\bar{\bar{T}}_s$.
2. Sa partie antisymétrique $\bar{\bar{T}}_a$.
3. Sa trace.
4. Son déterminant.

Rép. 3

1. Partie symétrique $\bar{\bar{T}}_s$ du tenseur $\bar{\bar{T}}$

$$\bar{\bar{\boldsymbol{T}}}_S = \frac{1}{2}\left(\bar{\bar{\boldsymbol{T}}} + {}^t\bar{\bar{\boldsymbol{T}}}\right) \quad \Rightarrow \quad (\boldsymbol{t_s})_{ij} = \frac{1}{2}\left(\boldsymbol{t}_{ij} + \boldsymbol{t}_{ji}\right)$$

$$\Rightarrow \quad \bar{\bar{T}}_s = \frac{1}{2}\left[\begin{pmatrix} 8 & 3 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 8 & -3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}\right] = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

2. Partie antisymétrique $\bar{\bar{T}}_a$ du tenseur $\bar{\bar{T}}$

$$\bar{\bar{\boldsymbol{T}}}_a = \frac{1}{2}\left(\bar{\bar{\boldsymbol{T}}} - {}^t\bar{\bar{\boldsymbol{T}}}\right) \quad \Rightarrow \quad (\boldsymbol{t_a})_{ij} = \frac{1}{2}\left(\boldsymbol{t}_{ij} - \boldsymbol{t}_{ji}\right)$$

$$\Rightarrow \quad \bar{\bar{T}}_a = \frac{1}{2}\left[\begin{pmatrix} 8 & 3 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} - \begin{pmatrix} 8 & -3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}\right] = \begin{pmatrix} 0 & 3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Trace du tenseur $\bar{\bar{T}}$

$$\boldsymbol{Tr}\left(\bar{\bar{\boldsymbol{T}}}\right) = \boldsymbol{t}_{ii} \quad \Rightarrow \quad Tr\left(\bar{\bar{T}}\right) = t_{11} + t_{22} + t_{33} = 8 - 4 - 4 = 0$$

Remarque : $Tr\left(\bar{\bar{T}}\right) = Tr\left(\bar{\bar{T}}_s\right) = Tr\left(\bar{\bar{T}}_a\right) = 0$

$\Rightarrow \quad \bar{\bar{T}}$ – Tenseur antisymétrique ($t_{ij} = -t_{ji}$ pour $i \neq j$).

4. Déterminant du tenseur $\bar{\bar{T}}$

$$\boldsymbol{det}\left(\bar{\bar{\boldsymbol{T}}}\right) = \begin{vmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} & \boldsymbol{t}_{13} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} & \boldsymbol{t}_{23} \\ \boldsymbol{t}_{31} & \boldsymbol{t}_{32} & \boldsymbol{t}_{33} \end{vmatrix}$$

$$\Rightarrow \quad det\left(\bar{\bar{T}}\right) = \begin{vmatrix} 8 & 3 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & -4 \end{vmatrix} = \begin{vmatrix} -4 & 0 \\ 0 & -4 \end{vmatrix} \times 8 - \begin{vmatrix} -3 & 0 \\ 0 & -4 \end{vmatrix} \times 3 + \begin{vmatrix} -3 & -4 \\ 0 & 0 \end{vmatrix} \times 0 = 92$$

Vérification :

$$\boldsymbol{det}\left(\bar{\bar{\boldsymbol{T}}}\right) = \frac{1}{6}\boldsymbol{\varepsilon}_{ijk}\boldsymbol{\epsilon}_{pqr}\boldsymbol{t}_{ip}\boldsymbol{t}_{jq}\boldsymbol{t}_{kr}$$

$$
\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix}
\epsilon_{123}\epsilon_{123}t_{11}t_{22}t_{33} + \epsilon_{123}\epsilon_{231}t_{12}t_{23}t_{31} + \epsilon_{123}\epsilon_{312}t_{13}t_{21}t_{32} \\
+ \\
\epsilon_{123}\epsilon_{132}t_{11}t_{23}t_{32} + \epsilon_{123}\epsilon_{213}t_{12}t_{21}t_{33} + \epsilon_{123}\epsilon_{321}t_{13}t_{22}t_{31} \\
+ \\
\epsilon_{231}\epsilon_{123}t_{21}t_{32}t_{13} + \epsilon_{231}\epsilon_{231}t_{22}t_{33}t_{11} + \epsilon_{231}\epsilon_{312}t_{23}t_{31}t_{12} \\
+ \\
\epsilon_{231}\epsilon_{132}t_{21}t_{33}t_{12} + \epsilon_{231}\epsilon_{213}t_{22}t_{31}t_{13} + \epsilon_{231}\epsilon_{321}t_{23}t_{32}t_{11} \\
+ \\
\epsilon_{312}\epsilon_{123}t_{31}t_{12}t_{23} + \epsilon_{312}\epsilon_{231}t_{32}t_{13}t_{21} + \epsilon_{312}\epsilon_{312}t_{33}t_{11}t_{22} \\
+ \\
\epsilon_{312}\epsilon_{132}t_{31}t_{13}t_{22} + \epsilon_{312}\epsilon_{213}t_{32}t_{11}t_{23} + \epsilon_{312}\epsilon_{321}t_{33}t_{12}t_{21} \\
+ \\
\epsilon_{132}\epsilon_{123}t_{11}t_{32}t_{23} + \epsilon_{132}\epsilon_{231}t_{12}t_{33}t_{21} + \epsilon_{132}\epsilon_{312}t_{13}t_{31}t_{22} \\
+ \\
\epsilon_{132}\epsilon_{132}t_{11}t_{33}t_{22} + \epsilon_{132}\epsilon_{213}t_{12}t_{31}t_{23} + \epsilon_{132}\epsilon_{321}t_{13}t_{32}t_{21} \\
+ \\
\epsilon_{213}\epsilon_{123}t_{21}t_{12}t_{33} + \epsilon_{213}\epsilon_{231}t_{22}t_{13}t_{31} + \epsilon_{213}\epsilon_{312}t_{23}t_{11}t_{32} \\
+ \\
\epsilon_{213}\epsilon_{132}t_{21}t_{13}t_{32} + \epsilon_{213}\epsilon_{213}t_{22}t_{11}t_{33} + \epsilon_{213}\epsilon_{321}t_{23}t_{12}t_{31} \\
+ \\
\epsilon_{321}\epsilon_{123}t_{31}t_{22}t_{13} + \epsilon_{321}\epsilon_{231}t_{32}t_{23}t_{11} + \epsilon_{321}\epsilon_{312}t_{33}t_{21}t_{12} \\
+ \\
\epsilon_{321}\epsilon_{132}t_{31}t_{23}t_{12} + \epsilon_{321}\epsilon_{213}t_{32}t_{21}t_{13} + \epsilon_{321}\epsilon_{321}t_{33}t_{22}t_{11}
\end{pmatrix}
$$

$$
\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix}
(+1)\times(+1)\times 8\times(-4)\times(-4) + (+1)\times(+1)\times 3\times 0\times 0 + (+1)\times(+1)\times 0\times(-3)\times 0 \\
+ \\
(+1)\times(-1)\times 8\times 0\times 0 + (+1)\times(-1)\times 3\times(-3)\times(-4) + (+1)\times(-1)\times 0\times(-4)\times 0 \\
+ \\
(+1)\times(+1)\times(-3)\times 0\times 0 + (+1)\times(+1)\times(-4)\times(-4)\times 8 + (+1)\times(+1)\times 0\times 0\times 3 \\
+ \\
(+1)\times(-1)\times(-3)\times(-4)\times 3 + (+1)\times(-1)\times(-4)\times 0\times 0 + (+1)\times(-1)\times 0\times 0\times 8 \\
+ \\
(+1)\times(+1)\times 0\times 3\times 0 + (+1)\times(+1)\times 0\times 0\times(-3) + (+1)\times(+1)\times(-4)\times 8\times(-4) \\
+ \\
(+1)\times(-1)\times 0\times 0\times(-4) + (+1)\times(-1)\times 0\times 8\times 0 + (+1)\times(-1)\times(-4)\times 3\times(-3) \\
+ \\
(-1)\times(+1)\times 8\times 0\times 0 + (-1)\times(+1)\times 3\times(-4)\times(-3) + (-1)\times(+1)\times 0\times 0\times(-4) \\
+ \\
(-1)\times(-1)\times 8\times(-4)\times(-4) + (-1)\times(-1)\times 3\times 0\times 0 + (-1)\times(-1)\times 0\times 0\times(-3) \\
+ \\
(-1)\times(+1)\times(-3)\times 3\times(-4) + (-1)\times(+1)\times(-4)\times 0\times 0 + (-1)\times(+1)\times 0\times 8\times 0 \\
+ \\
(-1)\times(+1)\times(-3)\times 0\times 0 + (-1)\times(+1)\times(-4)\times 8\times(-4) + (-1)\times(+1)\times 0\times 3\times 0 \\
+ \\
(-1)\times(+1)\times 0\times(-4)\times 0 + (-1)\times(+1)\times 0\times 0\times 8 + (-1)\times(+1)\times(-4)\times(-3)\times 3 \\
+ \\
(-1)\times(-1)\times 0\times 0\times 3 + (-1)\times(-1)\times 0\times(-3)\times 0 + (-1)\times(-1)\times(-4)(-4)\times 8
\end{pmatrix}
$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = \frac{1}{6}\begin{pmatrix} (+1)\times(+1)\times 8\times(-4)\times(-4) \\ + \\ (+1)\times(-1)\times 3\times(-3)\times(-4) \\ + \\ (+1)\times(+1)\times(-4)\times(-4)\times 8 \\ + \\ (+1)\times(-1)\times(-3)\times(-4)\times 3 \\ + \\ (+1)\times(+1)\times(-4)\times 8\times(-4) \\ + \\ (+1)\times(-1)\times(-4)\times 3\times(-3) \\ + \\ (-1)\times(+1)\times 3\times(-4)\times(-3) \\ + \\ (-1)\times(-1)\times 8\times(-4)\times(-4) \\ + \\ (-1)\times(+1)\times(-3)\times 3\times(-4) \\ + \\ (-1)\times(+1)\times(-4)\times 8\times(-4) \\ + \\ (-1)\times(+1)\times(-4)\times(-3)\times 3 \\ + \\ (-1)\times(-1)\times(-4)(-4)\times 8 \end{pmatrix} = \frac{1}{6}(92+92+92+92+92+92) = 92$$

$$\Rightarrow \quad det(\bar{\bar{T}}) = 92 \qquad Ok.$$

Ex. 4 Démontrer les identités suivantes :

1. $\left[Tr(\bar{\bar{M}})\right]^2 = M_{pp}M_{qq}$
2. $Tr(\bar{\bar{M}}^2) = M_{pq}M_{qp}$
3. $Tr(\bar{\bar{M}}\ {}^t\bar{\bar{M}}) = M_{pq}M_{pq} = Tr({}^t\bar{\bar{M}}\bar{\bar{M}})$
4. $\delta_{ij}V_j = V_i$
5. $\delta_{pq}V_pW_q = V_lW_l$
6. $\delta_{ii} = \delta_{ip}\delta_{pi} = \delta_{ip}\delta_{pq}\delta_{qi} = 3$

<u>Rép</u>. 4

1. $\left[\boldsymbol{Tr}(\bar{\bar{\boldsymbol{M}}})\right]^2 = \boldsymbol{M_{pp}M_{qq}}$

Sachant que : $Tr(\bar{\bar{M}}) = M_{ii}$

$$\Rightarrow \quad \left[Tr(\bar{\bar{M}})\right]^2 = (M_{ii})^2 = (M_{11}+M_{22}+M_{33})^2$$

$$\Rightarrow \quad \left[Tr(\bar{\bar{M}})\right]^2 = (M_{11}+M_{22}+M_{33})\times(M_{11}+M_{22}+M_{33})$$

$\Rightarrow \quad \left[Tr\left(\bar{\bar{M}}\right)\right]^2 = M_{11} \times (M_{11} + M_{22} + M_{33}) + M_{22} \times (M_{11} + M_{22} + M_{33}) + M_{33} \times (M_{11} + M_{22} + M_{33})$

$\Rightarrow \quad [Tr(M)]^2 = M_{11} \times \sum_{j=1}^{j=3} M_{jj} + M_{22} \times \sum_{j=1}^{j=3} M_{jj} + M_{33} \times \sum_{j=1}^{j=3} M_{jj}$

$\Rightarrow \quad \left[Tr\left(\bar{\bar{M}}\right)\right]^2 = \sum_{i=1}^{i=3} M_{ii} \times \sum_{j=1}^{j=3} M_{jj} = M_{ii} \times M_{jj} = M_{ii}M_{jj}$

$\Rightarrow \quad \left[Tr\left(\bar{\bar{M}}\right)\right]^2 = M_{pp}M_{qq}$

2. $\boldsymbol{Tr\left(\bar{\bar{M}}^2\right) = M_{pq}M_{qp}}$

Sachant que : $\bar{\bar{M}} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \quad \Rightarrow \quad \bar{\bar{M}}^2 = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$

$$\Rightarrow \quad \bar{\bar{M}}^2 = \begin{pmatrix} M_{11}M_{11} + M_{12}M_{21} + M_{13}M_{31} & M_{11}M_{12} + M_{12}M_{22} + M_{13}M_{32} & M_{11}M_{13} + M_{12}M_{23} + M_{13}M_{33} \\ M_{21}M_{11} + M_{22}M_{21} + M_{23}M_{31} & M_{21}M_{12} + M_{22}M_{22} + M_{23}M_{32} & M_{21}M_{13} + M_{22}M_{23} + M_{23}M_{33} \\ M_{31}M_{11} + M_{32}M_{21} + M_{33}M_{31} & M_{31}M_{12} + M_{32}M_{22} + M_{33}M_{32} & M_{31}M_{13} + M_{32}M_{23} + M_{33}M_{33} \end{pmatrix}$$

$$\Rightarrow \quad Tr\left(\bar{\bar{M}}^2\right) = \begin{pmatrix} M_{11}M_{11} + M_{12}M_{21} + M_{13}M_{31} \\ + \\ M_{21}M_{12} + M_{22}M_{22} + M_{23}M_{32} \\ + \\ M_{31}M_{13} + M_{32}M_{23} + M_{33}M_{33} \end{pmatrix} = \sum_{j=1}^{j=3} M_{1j}M_{j1} + \sum_{j=1}^{j=3} M_{2j}M_{j2} + \sum_{j=1}^{j=3} M_{3j}M_{j3}$$

$\Rightarrow \quad Tr\left(\bar{\bar{M}}^2\right) = \sum_{i=1}^{i=3}\sum_{j=1}^{j=3} M_{ij}M_{ji} = M_{ij}M_{ji}$

$\Rightarrow \quad Tr\left(\bar{\bar{M}}^2\right) = M_{pq}M_{qp}$

3. $\boldsymbol{Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = M_{pq}M_{pq}}$

Sachant que : $\bar{\bar{M}} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \quad \Rightarrow \quad {}^t\bar{\bar{M}} = \begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$

$\Rightarrow \quad \bar{\bar{M}}\ {}^t\bar{\bar{M}} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}$

$$\Rightarrow \quad \bar{\bar{M}}\ {}^t\bar{\bar{M}} = \begin{pmatrix} M_{11}M_{11} + M_{12}M_{12} + M_{13}M_{13} & M_{11}M_{21} + M_{12}M_{22} + M_{13}M_{23} & M_{11}M_{31} + M_{12}M_{32} + M_{13}M_{33} \\ M_{21}M_{11} + M_{22}M_{12} + M_{23}M_{13} & M_{21}M_{21} + M_{22}M_{22} + M_{23}M_{23} & M_{21}M_{31} + M_{22}M_{32} + M_{23}M_{33} \\ M_{31}M_{11} + M_{32}M_{12} + M_{33}M_{13} & M_{31}M_{21} + M_{32}M_{22} + M_{33}M_{23} & M_{31}M_{31} + M_{32}M_{32} + M_{33}M_{33} \end{pmatrix}$$

$$\Rightarrow \quad Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = \begin{pmatrix} M_{11}M_{11} + M_{12}M_{12} + M_{13}M_{13} + M_{21}M_{21} + M_{22}M_{22} + M_{23}M_{23} \\ + \\ M_{21}M_{21} + M_{22}M_{22} + M_{23}M_{23} \\ + \\ M_{31}M_{31} + M_{32}M_{32} + M_{33}M_{33} \end{pmatrix}$$

$\Rightarrow \quad Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = \sum_{j=1}^{j=3} M_{1j}M_{1j} + \sum_{j=1}^{j=3} M_{2j}M_{2j} + \sum_{j=1}^{j=3} M_{3j}M_{3j}$

$\Rightarrow \quad Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = \sum_{i=1}^{i=3}\sum_{j=1}^{j=3} M_{ij}M_{ij} = M_{ij}M_{ij}$

$\Rightarrow \quad Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = M_{pq}M_{pq}$

<u>Remarque</u> : $Tr\left(\bar{\bar{M}}\ {}^t\bar{\bar{M}}\right) = Tr\left({}^t\bar{\bar{M}}\bar{\bar{M}}\right)$

4. $\boldsymbol{\delta_{ij}V_j = V_i}$

Sachant que : $\delta_{ij} = \begin{cases} 1\ si\ i = j \\ 0\ si\ i \neq j \end{cases}$

$\Rightarrow \quad \delta_{ij}V_j = \delta_{i1}V_1 + \delta_{i2}V_2 + \delta_{i3}V_3 \quad \Rightarrow \begin{cases} \delta_{11}V_1 + \cancel{\delta_{12}}V_2 + \cancel{\delta_{13}}V_3 = V_1 & (i = 1) \\ \cancel{\delta_{21}}V_1 + \delta_{22}V_2 + \cancel{\delta_{23}}V_3 = V_2 & (i = 2) \\ \cancel{\delta_{31}}V_1 + \cancel{\delta_{32}}V_2 + \delta_{33}V_3 = V_3 & (i = 3) \end{cases}$

$\Rightarrow \quad \delta_{11}V_1 + \delta_{22}V_2 + \delta_{33}V_3 = V_1 + V_2 + V_3$

$\Rightarrow \quad \delta_{ij}V_j = V_i$

5. $\boldsymbol{\delta_{pq}V_pW_q = V_lW_l}$

Sachant que : $\delta_{pq}V_q = V_p$ (Voir Ex. 4)

$\Rightarrow \quad \delta_{pq}V_pW_q = \left(\delta_{pq}V_p\right)W_q = V_qW_q$

$\Rightarrow \quad \delta_{pq}V_pW_q = V_lW_l$

6. $\boldsymbol{\delta_{ii} = \delta_{ip}\delta_{pi} = \delta_{ip}\delta_{pq}\delta_{qi} = 3}$

Sachant que : $\delta_{ij} = \delta_{ji}$ (Le tenseur unité est symétrique)

$\Rightarrow \quad \delta_{ij}\delta_{ij} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3$

$\Rightarrow \quad \delta_{ip}\delta_{pq}\delta_{qi} = \left(\delta_{ip}\delta_{pq}\right)\delta_{qi} = \delta_{iq}\delta_{qi} = \delta_{ii} = 3$

Ex. 5 Soit la fonction scalaire suivante : $\varphi(\bar{x}) = \bar{x}^2$

On demande de calculer :

1. $\overline{grad}(\varphi) = \bar{V}$
2. $\Delta(\varphi) = div(\bar{V})$
3. $\overline{rot}(\bar{V})$
4. $\overline{\overline{grad}}(\bar{V})$

<u>Rép</u>. 5

- Sachant que : $\bar{x} = x_i\bar{e}_i = x_1\bar{e}_1 + x_2\bar{e}_2 + x_3\bar{e}_3$ & $\bar{e}_i \cdot \bar{e}_j = \begin{cases} 1\ si\ i = j \\ 0\ si\ i \neq j \end{cases}$

$\Rightarrow \quad \varphi(\bar{x}) = \bar{x}^2 = (x_i\bar{e}_i)^2 = (x_1\bar{e}_1 + x_2\bar{e}_2 + x_3\bar{e}_3)^2 = x_1^2 + x_2^2 + x_3^2$

1. $\boldsymbol{\overline{grad}(\varphi)} = \frac{\partial\varphi}{\partial x_i}\boldsymbol{\bar{e}_i} = \frac{\partial\varphi}{\partial x_1}\boldsymbol{\bar{e}_1} + \frac{\partial\varphi}{\partial x_2}\boldsymbol{\bar{e}_2} + \frac{\partial\varphi}{\partial x_3}\boldsymbol{\bar{e}_3} = 2x_1\bar{e}_1 + 2x_2\bar{e}_2 + 2x_3\bar{e}_3 = 2\bar{x} = \begin{Bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{Bmatrix} \ / \left(\bar{e}_1, \bar{e}_2, \bar{e}_3\right)$

2. $\Delta(\boldsymbol{\varphi}) = \frac{\partial^2\varphi}{\partial x_i^2} = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} = 2 + 2 + 2 = 6$

Posons : $\bar{V} = \overline{grad}(\varphi) = 2\bar{x} \quad \Rightarrow \quad \boldsymbol{div(\bar{V})} = \frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} = 2 + 2 + 2 = 6$

$$\Rightarrow \quad \boldsymbol{div\left[\overline{grad}(\varphi)\right]} = \Delta(\boldsymbol{\varphi})$$

3. $\boldsymbol{\overline{rot}(\bar{V})} = {}^t\left\{\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3} \ , \ \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1} \ , \ \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right\} = {}^t\{0 \ , \ 0 \ , \ 0\} = \bar{0}$

4. $\boldsymbol{\overline{\overline{grad}}(\bar{V})} = \frac{\partial V_i}{\partial x_j}\bar{e}_i \otimes \bar{e}_j = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \frac{\partial V_1}{\partial x_3} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_3} \\ \frac{\partial V_3}{\partial x_1} & \frac{\partial V_3}{\partial x_2} & \frac{\partial V_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2\bar{\bar{\delta}}$

Ex. 6 Soit les deux vecteurs suivants : $\bar{U} = \bar{x}$ et $\bar{V} = 2\bar{x}$

On demande de calculer :

1. $\overline{\overline{grad}}(\bar{V}) = \bar{\bar{M}}$
2. $\bar{U} \cdot \overline{\overline{grad}}(\bar{V})$
3. $\overline{\overline{grad}}(\bar{V}) \cdot \bar{U}$
4. $\overline{div}(\bar{\bar{M}})$
5. $\Delta(\bar{V})$

Rép. 6

- $\bar{U} = \bar{x} = x_i\bar{e}_i = x_1\bar{e}_1 + x_2\bar{e}_2 + x_3\bar{e}_3 = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad / \ (\bar{e}_1, \bar{e}_2, \bar{e}_3)$
- $\bar{V} = 2\bar{x} = 2x_i\bar{e}_i = 2x_1\bar{e}_1 + 2x_2\bar{e}_2 + 2x_3\bar{e}_3 = \begin{Bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{Bmatrix} \quad / \ (\bar{e}_1, \bar{e}_2, \bar{e}_3)$

1. $\boldsymbol{\overline{\overline{grad}}(\bar{V})} = \frac{\partial V_i}{\partial x_j}\bar{e}_i \otimes \bar{e}_j = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \frac{\partial V_1}{\partial x_3} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_3} \\ \frac{\partial V_3}{\partial x_1} & \frac{\partial V_3}{\partial x_2} & \frac{\partial V_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2\bar{\bar{\delta}} = \bar{\bar{M}}$

2. $\boldsymbol{\bar{U} \cdot \overline{\overline{grad}}(\bar{V})} = \boldsymbol{\bar{U}} \cdot \bar{\bar{M}} = {}^t\{x_1 \ , \ x_2 \ , \ x_3\}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{Bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{Bmatrix} = 2\bar{x}$

3. $\boldsymbol{\overline{\overline{grad}}(\bar{V}) \cdot \bar{U}} = \bar{\bar{M}} \cdot \boldsymbol{\bar{U}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{Bmatrix} = 2\bar{x}$

$$\Rightarrow \quad \boldsymbol{\bar{U} \cdot \overline{\overline{grad}}(\bar{V}) = \overline{\overline{grad}}(\bar{V}) \cdot \bar{U}}$$

4. $\overline{\boldsymbol{div}}(\overline{\overline{\boldsymbol{M}}}) = \frac{\partial M_{ij}}{\partial x_j}\bar{\boldsymbol{e}}_i = \begin{Bmatrix} \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + \frac{\partial M_{13}}{\partial x_3} \\ \frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{23}}{\partial x_3} \\ \frac{\partial M_{31}}{\partial x_1} + \frac{\partial M_{32}}{\partial x_2} + \frac{\partial M_{33}}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \bar{0}$

5. $\Delta(\overline{\boldsymbol{V}}) = \frac{\partial^2 V_i}{\partial x_i^2}\bar{\boldsymbol{e}}_i = \begin{Bmatrix} \frac{\partial^2 V_1}{\partial x_1^2} + \frac{\partial^2 V_1}{\partial x_2^2} + \frac{\partial^2 V_1}{\partial x_3^2} \\ \frac{\partial^2 V_2}{\partial x_1^2} + \frac{\partial^2 V_2}{\partial x_2^2} + \frac{\partial^2 V_2}{\partial x_3^2} \\ \frac{\partial^2 V_3}{\partial x_1^2} + \frac{\partial^2 V_1}{\partial x_2^2} + \frac{\partial^2 V_1}{\partial x_3^2} \end{Bmatrix} = \begin{Bmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \bar{0}$

Ex. 7 Soit les tenseurs suivants : $\overline{\overline{M}} = \bar{x}\otimes\bar{e}_1$ et $\overline{\overline{N}} = \bar{e}_1\otimes\bar{x}$

On demande de calculer :

1. $div(\overline{\overline{M}})$
2. $div(\overline{\overline{N}})$

Rép. 7

- Sachant que : $\bar{x} = x_i\bar{e}_i = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ & $\bar{e}_1 = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ / $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$

$$\Rightarrow \quad \overline{\overline{\boldsymbol{M}}} = \bar{\boldsymbol{x}}\otimes\bar{\boldsymbol{e}}_1 = \begin{bmatrix} \boldsymbol{x_1e_1} & \boldsymbol{x_1e_2} & \boldsymbol{x_1e_3} \\ \boldsymbol{x_2e_1} & \boldsymbol{x_2e_2} & \boldsymbol{x_2e_3} \\ \boldsymbol{x_3e_1} & \boldsymbol{x_3e_2} & \boldsymbol{x_3e_3} \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \overline{\overline{\boldsymbol{N}}} = \bar{\boldsymbol{e}}_1\otimes\bar{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{e_1x_1} & \boldsymbol{e_1x_2} & \boldsymbol{e_1x_3} \\ \boldsymbol{e_2x_1} & \boldsymbol{e_2x_2} & \boldsymbol{e_2x_3} \\ \boldsymbol{e_3x_1} & \boldsymbol{e_3x_2} & \boldsymbol{e_3x_3} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. $\overline{\boldsymbol{div}}(\overline{\overline{\boldsymbol{M}}}) = \frac{\partial M_{ij}}{\partial x_j}\bar{\boldsymbol{e}}_i = \begin{Bmatrix} \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + \frac{\partial M_{13}}{\partial x_3} \\ \frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} + \frac{\partial M_{23}}{\partial x_3} \\ \frac{\partial M_{31}}{\partial x_1} + \frac{\partial M_{32}}{\partial x_2} + \frac{\partial M_{33}}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} 1+0+0 \\ 0+0+0 \\ 0+0+0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \bar{e}_1$

2. $\overline{\boldsymbol{div}}(\overline{\overline{\boldsymbol{N}}}) = \frac{\partial N_{ij}}{\partial x_j}\bar{\boldsymbol{e}}_i = \begin{Bmatrix} \frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + \frac{\partial N_{13}}{\partial x_3} \\ \frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + \frac{\partial N_{23}}{\partial x_3} \\ \frac{\partial N_{31}}{\partial x_1} + \frac{\partial N_{32}}{\partial x_2} + \frac{\partial N_{33}}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} 1+1+1 \\ 0+0+0 \\ 0+0+0 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 0 \\ 0 \end{Bmatrix} = 3\bar{e}_1$

Ex. 8 Posons : $\varphi(\bar{x}) = \bar{x}^2$
$\bar{U} = \bar{x}$
$\bar{V} = 2\bar{x}$

On demande de calculer :
1. $div(\varphi\bar{V})$
2. $\overline{div}(\bar{U}\otimes\bar{V})$
3. $\overline{rot}[\overline{rot}(\bar{V})]$

Rép. 8

1. $\boldsymbol{div(\varphi\bar{V}) = \varphi\overline{div}(\bar{V}) + \bar{V}\cdot\overline{grad}(\varphi)}$

Sachant que :

- $\boldsymbol{div(\bar{V}) = \frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}} = 2 + 2 + 2 = 6$
- $\boldsymbol{\overline{grad}(\varphi) = \bar{V} = \frac{\partial \varphi}{\partial x_i}\bar{e}_i = \frac{\partial \varphi}{\partial x_1}\bar{e}_1 + \frac{\partial \varphi}{\partial x_2}\bar{e}_2 + \frac{\partial \varphi}{\partial x_3}\bar{e}_3} = 2x_1\bar{e}_1 + 2x_2\bar{e}_2 + 2x_3\bar{e}_3 = 2\bar{x}$

$\Rightarrow \quad div(\varphi\bar{V}) = (\bar{x}^2)(6) + (2\bar{x})(2\bar{x}) = 6\bar{x}^2 + 4\bar{x}^2 = 10\bar{x}^2$

2. $\boldsymbol{\overline{div}(\bar{U}\otimes\bar{V}) = \overline{\overline{grad}}(\bar{U})\cdot\bar{V} + div(\bar{V})\bar{U}}$

Sachant que :

- $\boldsymbol{div(\bar{V}) = \frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}} = 2 + 2 + 2 = 6$
- $\boldsymbol{\overline{\overline{grad}}(\bar{U}) = \frac{\partial U_i}{\partial x_j}\bar{e}_i\otimes\bar{e}_j} = \begin{bmatrix} \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} & \frac{\partial U_1}{\partial x_3} \\ \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial x_3} \\ \frac{\partial U_3}{\partial x_1} & \frac{\partial U_3}{\partial x_2} & \frac{\partial U_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \bar{\bar{\delta}}$

$$\Rightarrow \quad \overline{div}(\bar{U}\otimes\bar{V}) = {}^t\bar{V}\cdot\bar{\bar{\delta}} + div(\bar{V})\bar{U}$$

$$= {}^t\{2\,,\ 2\,,\ 2\}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 6\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 6 \\ 6 \\ 6 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 8 \\ 8 \end{Bmatrix} \qquad / \,(\bar{e}_1, \bar{e}_2, \bar{e}_3\,)$$

$$= 8\bar{x}$$

<u>Vérification</u> :

$$\bar{W} = \bar{U}\otimes\bar{V} = \begin{bmatrix} U_1V_1 & U_1V_2 & U_1V_3 \\ U_2V_1 & U_2V_2 & U_2V_3 \\ U_3V_1 & U_3V_2 & U_3V_3 \end{bmatrix} = \begin{bmatrix} x_1\times 2x_1 & x_1\times 2x_2 & x_1\times 2x_3 \\ x_2\times 2x_1 & x_2\times 2x_2 & x_2\times 2x_3 \\ x_3\times 2x_1 & x_3\times 2x_2 & x_3\times 2x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1^2 & 2x_1x_2 & 2x_1x_3 \\ 2x_1x_2 & 2x_2^2 & 2x_2x_3 \\ 2x_1x_3 & 2x_2x_3 & 2x_3^2 \end{bmatrix} = 2\begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix}$$

$$\Rightarrow \quad \overline{div}(\bar{W}) = \overline{div}(\bar{U}\otimes\bar{V}) = \begin{Bmatrix} \dfrac{\partial W_{11}}{\partial x_1} + \dfrac{\partial W_{12}}{\partial x_2} + \dfrac{\partial W_{13}}{\partial x_3} \\ \dfrac{\partial W_{21}}{\partial x_1} + \dfrac{\partial W_{22}}{\partial x_2} + \dfrac{\partial W_{23}}{\partial x_3} \\ \dfrac{\partial W_{31}}{\partial x_1} + \dfrac{\partial W_{32}}{\partial x_2} + \dfrac{\partial W_{33}}{\partial x_3} \end{Bmatrix} = 2\begin{Bmatrix} 2x_1 + x_1 + x_1 \\ x_2 + 2x_2 + x_2 \\ x_3 + x_3 + 2x_3 \end{Bmatrix} = 8\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 8\bar{x}$$

3. $\overline{\boldsymbol{rot}}[\overline{\boldsymbol{rot}}(\bar{\boldsymbol{V}})] = \overline{\boldsymbol{grad}}[\boldsymbol{div}(\bar{\boldsymbol{V}})] - \bar{\Delta}(\bar{\boldsymbol{V}})$

Sachant que :

- $\boldsymbol{div}(\bar{\boldsymbol{V}}) = \dfrac{\partial \boldsymbol{V_i}}{\partial \boldsymbol{x_i}} = \dfrac{\partial \boldsymbol{V_1}}{\partial \boldsymbol{x_1}} + \dfrac{\partial \boldsymbol{V_2}}{\partial \boldsymbol{x_2}} + \dfrac{\partial \boldsymbol{V_3}}{\partial \boldsymbol{x_3}} = 2 + 2 + 2 = 6$

  $\Rightarrow \quad \overline{grad}[div(\bar{V})] = \overline{grad}(6) = {}^t\{0\,,0\,,0\} = \bar{0}$

- $$\bar{\Delta}(\bar{\boldsymbol{V}}) = \frac{\partial^2 \boldsymbol{V_i}}{\partial \boldsymbol{x_j^2}}\bar{\boldsymbol{e}}_{\boldsymbol{i}} = \begin{Bmatrix} \dfrac{\partial^2 \boldsymbol{V_1}}{\partial \boldsymbol{x_1^2}} + \dfrac{\partial^2 \boldsymbol{V_1}}{\partial \boldsymbol{x_2^2}} + \dfrac{\partial^2 \boldsymbol{V_1}}{\partial \boldsymbol{x_3^2}} \\ \dfrac{\partial^2 \boldsymbol{V_2}}{\partial \boldsymbol{x_1^2}} + \dfrac{\partial^2 \boldsymbol{V_2}}{\partial \boldsymbol{x_2^2}} + \dfrac{\partial^2 \boldsymbol{V_2}}{\partial \boldsymbol{x_3^2}} \\ \dfrac{\partial^2 \boldsymbol{V_3}}{\partial \boldsymbol{x_1^2}} + \dfrac{\partial^2 \boldsymbol{V_3}}{\partial \boldsymbol{x_2^2}} + \dfrac{\partial^2 \boldsymbol{V_3}}{\partial \boldsymbol{x_3^2}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \bar{0}$$

$\Rightarrow \quad \overline{rot}[\overline{rot}(\bar{V})] = \bar{0} + \bar{0} = \bar{0}$