Chapter III : Stress Analysis

- IV. Stresses in 3 dimensions
- The stress vector
- Stresses in 1 dimension
- Stresses in 2 dimensions
- Stress plane law transformation
- Principal stresses; maximum shearing stress

IV. Stresses in 3 dimensions:

IV.1 Stresses under general loading conditions; stress components:

Let we consider a body subjected to several loads P_1, P_2, etc. (Fig.28).

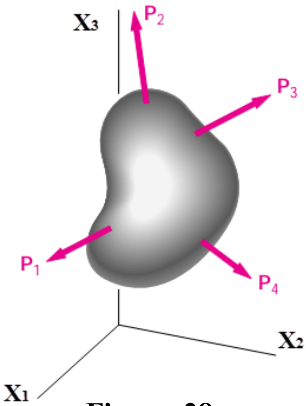

Figure .28.

To understand the stress condition created by these loads at a point Q of the body, we will first cut through a fictitious section through Q, using a plane parallel to the X_1X_2 plane.

The part of the body at the left of the section is subjected to some of the original loads, the normal and shear loads distributed over the section. We denote by ΔF_{X2} and ΔV_{X2} respectively the normal and shear loads acting on a small surface ΔA surrounding the point Q (Fig.29a). Note that the subscript X_2 is used to indicate that the forces ΔF_{X2} and ΔV_{X2} act on a surface perpendicular to the X_2 axis.

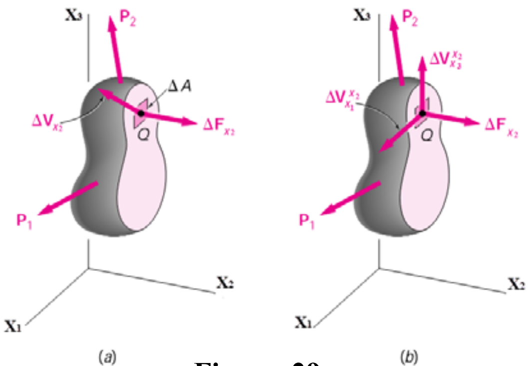

Figure .29.

Although the normal load ΔF_{X2} has a well-defined direction, the shear load ΔV_{X2} can have any direction in the plane of the section. We therefore solve ΔV_{X2} in two load components, ΔV_{X1}^{X2} and ΔV_{X3}^{X2} in directions parallel to X_2X_1 and X_2X_3, respectively (Fig. 29b). Now dividing the magnitude of each load by the area ΔA, we define the three stress components as shown in Fig.30. and Fig.31. :

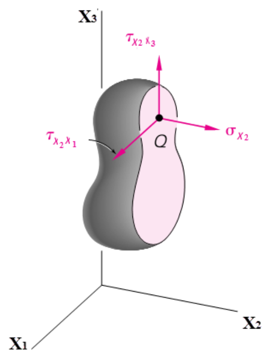

Figure .30.

In Fig.31., we notice that there is three principal facets denoted ①,② and ③ ;

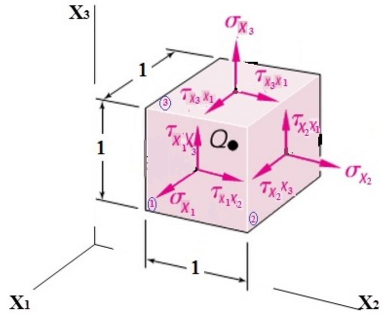

Figure .31.

$\sigma_{X2} = \frac{\Delta F\mathbf{X2}}{\Delta A}$ and : $\tau_{X2X1} = \frac{\Delta V_{X1}^{X2}}{\Delta A}$; $\tau_{X2X3} = \frac{\Delta V_{X3}^{X2}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_2 axis ($\vec{e}_2$).

The stress vector $\vec{T}_{\vec{e}2}$ will be composed with a normal stress σ_{X2} and two shear stresses τ_{X2X1} , τ_{X2X3}, noted as :

$$\vec{T}_{\vec{e}2} = \begin{Bmatrix} \sigma_{X2} \\ \tau_{X2X1} \\ \tau_{X2X3} \end{Bmatrix};$$

In the same way, we choose a plane passing through the point Q perpendicular to X_1 illustrated in Fig.32.

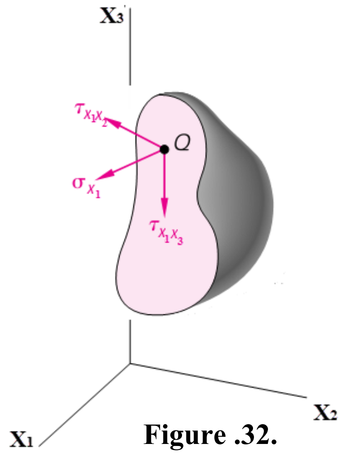

Figure .32.

$\sigma_{X1} = \frac{\Delta F\mathbf{X1}}{\Delta A}$ and : $\tau_{X1X2} = \frac{\Delta V_{X2}^{X1}}{\Delta A}$; $\tau_{X1X3} = \frac{\Delta V_{X3}^{X1}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_1 axis ($\vec{e}_1$).

The stress vector $\vec{T}_{\vec{e}1}$ will be composed with a normal stress σ_{X1} and two shear stresses τ_{X1X2} , τ_{X1X3}, noted as :

$$\vec{T}_{\vec{e}1} = \begin{Bmatrix} \sigma_{X1} \\ \tau_{X1X2} \\ \tau_{X1X3} \end{Bmatrix};$$

and so for the last axis X_3.

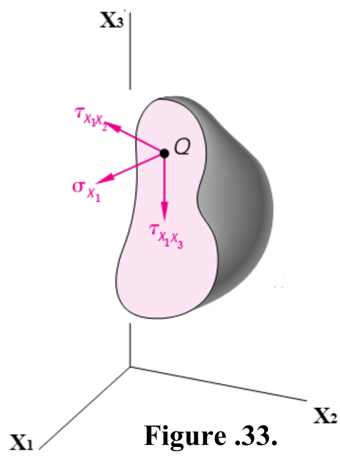

Figure .33.

$\sigma_{X3} = \frac{\Delta F\mathbf{X3}}{\Delta A}$ and : $\tau_{X3X1} = \frac{\Delta V_{X1}^{X3}}{\Delta A}$; $\tau_{X3X2} = \frac{\Delta V_{X2}^{X3}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_3 axis ($\vec{e}_3$).

The stress vector $\vec{T}_{\vec{e}3}$ will be composed with a normal stress σ_{X3} and two shear stresses τ_{X3X2} , τ_{X3X1}, noted as :

$$\vec{T}_{\vec{e}3} = \begin{Bmatrix} \sigma_{X3} \\ \tau_{X3X2} \\ \tau_{X3X1} \end{Bmatrix};$$

Finally, we can say that to define the state of stress σ_{ij} in the point Q, we must consider an elementary cube of unit as edge subjected to the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{X1} & \tau_{X1X2} & \tau_{X1X3} \\ \tau_{X2X1} & \sigma_{X2} & \tau_{X2X3} \\ \tau_{X3X1} & \tau_{X3X2} & \sigma_{X3} \end{bmatrix}$$

With the diagonal stresses $\sigma_{X1}, \sigma_{X2}, \sigma_{X3}$ they are called normal stresses, and the other stresses $\tau_{X1X2}, \tau_{X1X3}, \tau_{X1X2}, \tau_{X2X1}, \tau_{X2X3}, \tau_{X3X1};\ \tau_{X3X2}$ they are called as "shear" stresses.

The stress tensor σ_{ij} is called as "3D Cauchy stress tensor" at point Q, the tensor is symmetric, i.e.;

$$\tau_{X1X2} = \tau_{X2X1}, \tau_{X1X3} = \tau_{X3X1}, \tau_{X2X3} = \tau_{X3X2}.$$

The component σ_{ij} of the Cauchy stress tensor, it represents the component along the direction $\vec{e}_i$ of the vector stress on the facet of normal $\vec{e}_j$ [1].

IV.2 The stress vector in 3D:

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $\vec{T}_{\vec{n}}$ on a plane of arbitrary orientation (facet) at that point by:

$$\vec{T}_{\vec{n}} = \begin{Bmatrix} \sigma_n \\ \tau_1 \\ \tau_2 \end{Bmatrix}$$

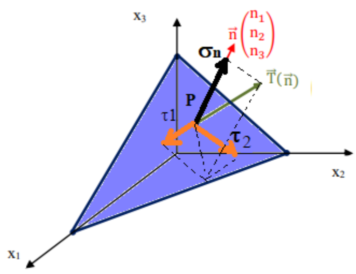

Figure .34.

But in general case the stresses state is defined by an elementary cube with three facets, for each facet we have a stress vector and an elementary direction (Fig.35.);

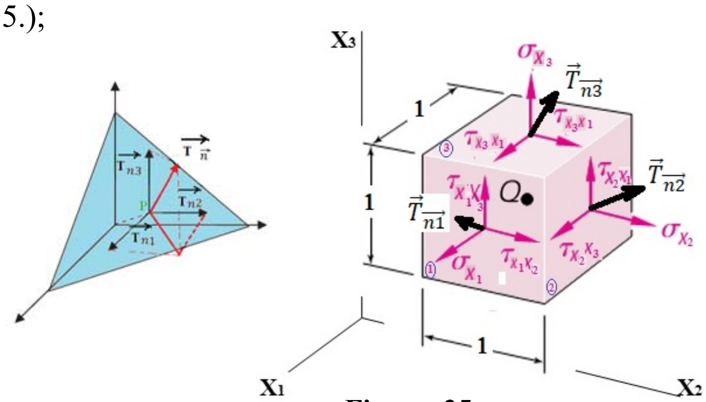

Figure .35.

The vector $\vec{T}_{\vec{n}}$ depends linearly with $\vec{n}$, therefore exists a linear application, moving from $\vec{T}_{\vec{n}}$ to $\vec{n}$ with :

$\vec{T}_{\vec{n}} = \sigma.\vec{n}$ ➔

$$\begin{Bmatrix} T_{\vec{n}1} \\ T_{\vec{n}2} \\ T_{\vec{n}3} \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11}n_1 + \tau_{12}n_2 + \tau_{13}n_3 \\ \tau_{21}n_1 + \sigma_{22}n_2 + \tau_{23}n_3 \\ \tau_{31}n_1 + \tau_{32}n_2 + \sigma_{33}n_3 \end{Bmatrix}$$

The component σ_{ij} of the Cauchy stress tensor, it represents the component along the direction $\vec{e}_i$ of the vector stress on the facet of normal $\vec{e}_j$ [1].

IV. 3D Stress transformation law:

At the point P let the rectangular Cartesian coordinate systems $(O, \vec{X}_1, \vec{X}_2, \vec{X}_3)$ et $(O, \vec{x}'_1, \vec{x}'_2, \vec{x}'_3)$ of Fig.36. be related to one another by the table of direction cosines:

	$\vec{x}'_1$	$\vec{x}'_2$	$\vec{x}'_3$
$\vec{X}_1$	a_{11}	a_{12}	a_{13}
$\vec{X}_2$	a_{21}	a_{22}	a_{23}
$\vec{X}_3$	a_{31}	a_{32}	a_{33}

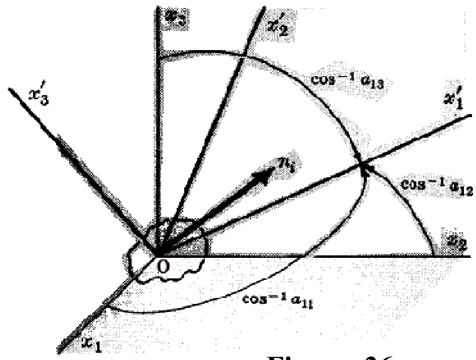

Figure .36.

The stress vector in the two bases can be written: $\vec{T}'_{\vec{n}} = a_{ij}\vec{T}_{\vec{n}}$.

$$\begin{Bmatrix} T'_{n1} \\ T'_{n2} \\ T'_{n3} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} T_{n1} \\ T_{n2} \\ T_{n3} \end{Bmatrix}$$

⇨ The stress tensor components in the two systems are related by : $\sigma'_{ij} = a_{ip}\,\sigma_{pq}a_{qj}$ ➔ $\sigma' = A\,\sigma\,A^T$

$$\begin{bmatrix} \sigma'_{11} & \tau'_{12} & \tau'_{13} \\ \tau'_{21} & \sigma'_{22} & \tau'_{23} \\ \tau'_{31} & \tau'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

IV.3 Principal stresses, stress invariants, stress ellipsoid :

By using the «Analytical method "; the symmetry of the stress tensor means that it has three eigenvalues $\sigma_I, \sigma_{II}, \sigma_{III}$, real, called principal stresses. By taking these three vectors (principal directions) as a coordinate system (principal coordinate system) (Figure .37.), the stress tensor becomes diagonal.

- Any coordinate system :

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

- Principal coordinate system:

$$\sigma^*_{ij} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

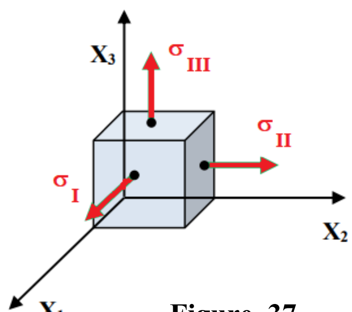

Figure .37.

The knowledge of principal stresses is very important and essentially concerns the calculation of structures in RDM. An important example is the application of the criterion for the appearance of the rupture.

And We can find a particle containing the point (P) and oriented in such a way that these facets will only be subject to the normal stresses (principal stresses) in exclusion of all shear stresses.

As conclusion: we find a particle oriented as the stress vector of the facets is the same as the normal-principal- stresses.

$\overrightarrow{T^*}_{\vec{n}} = \sigma_{ij}.\vec{n}_j = \sigma_i.\vec{n}_i$ with $\vec{n}_i = \delta_{ij}.\vec{n}_j$

Than : $(\sigma_{ij} - \delta_{ij}.\sigma_i)\,\vec{n}_j = \vec{0}$

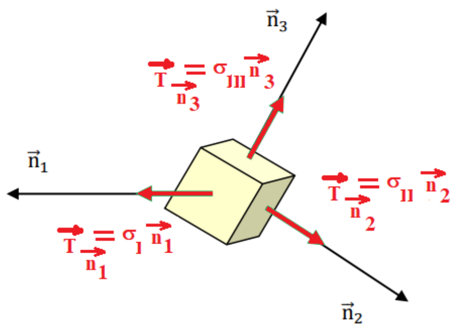

Figure .38.

For $\vec{n}_j = \vec{0}$ the solution is trivial, and since $(\sigma_{ij} - \delta_{ij}.\sigma_i)$ is a matrix where the unknowns are the principal stresses σ_i; this allows us to say that its determinant must be zero.

$$\rightarrow \begin{vmatrix} \sigma_{11} - \sigma & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} - \sigma & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} - \sigma \end{vmatrix} = 0,$$

We will obtain a polynomial equation of order three called the characteristic equation:

$$\sigma^3 - I\,\sigma^2 + II\,\sigma - III = 0$$

In convention, the values of the principal constraints are ordered as follows: $\sigma_I > \sigma_{II} > \sigma_{III}$.

Where I, II and III are called respectively the first, second and third stress invariant, with:

$$I = trace(\sigma_{ij}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$II = \frac{1}{2}(\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \tau_{12}^2 - \tau_{23}^2 - \tau_{13}^2$$

$$III = |\sigma_{ij}| = \det(\sigma_{ij}) = \sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{12}\tau_{23}\tau_{13} - \tau_{12}^2\sigma_{33} - \tau_{23}^2\sigma_{11} - \tau_{13}^2\sigma_{22}$$

The three roots of the characteristic equation are the principal stress values; each value of the principal stresses is associated with a principal direction for which the direction cosines are the solutions of the equations:

$$(\sigma_{ij} - \delta_{ij}.\sigma_i)\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \vec{0}$$

In the principal stress space where the axes are the principal directions, the stress vector will have the following three components:

$$\overrightarrow{T^*}_{\vec{n}} = \begin{Bmatrix} T^*_{n1} = \sigma_I.n_1 \\ T^*_{n2} = \sigma_{II}.n_2 \\ T^*_{n3} = \sigma_{III}.n_3 \end{Bmatrix}$$

We take into consideration the orthogonality condition: $n_i.n_i = 1$ or: $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$,

The stress vector must satisfy:

$$\frac{(T_{n1})^2}{(\sigma_I)^2} + \frac{(T_{n2})^2}{(\sigma_{II})^2} + \frac{(T_{n3})^2}{(\sigma_{III})^2} = 1$$

This equation is known as the Lamé's stress ellipsoid **[3]**.

Maximum shear stress:

The equation: $\tau_{max} = \frac{(\sigma_I - \sigma_{III})}{2}$ gives the maximum shear value, which is equal to half the difference of the largest and smallest principal stresses.

IV.3 MOHR's circles for 3D stress state:

Where the state of stress at a point P is given by the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

Each line represents the state of stresses on each facet of elementary cube as showed which will be described in each circle.

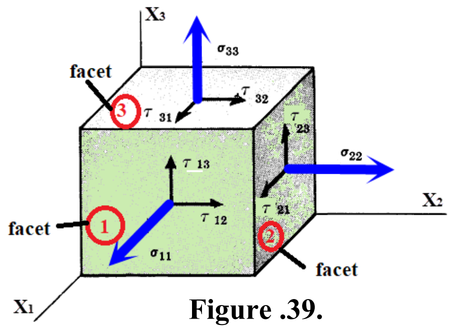

Figure .39.

The stress vector has normal and shear components whose magnitudes satisfy the equations:

$$\sigma_N = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_I n_3^2$$

And

$$\sigma_N^2 + \tau^2 = \sigma_I{}^2 n_1^2 + \sigma_{II}{}^2 n_2^2 + \sigma_{III}{}^2 n_3^2;$$

Combining these two expressions with the orthogonality condition: $n_i n_j = 1$ and solving for the direction cosine n_i, results in the equations:

$$(n_1)^2 = \frac{(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\tau)^2}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})};$$

$$(n_2)^2 = \frac{(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\tau)^2}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)};$$

$$(n_3)^2 = \frac{(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\tau)^2}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})};$$

These equations serve as the basis for Mohr's stress circles, as discussed in the "stress plane "section, for which the σ_n axis is the abscissa, and the τ axis is the ordinate.

To develop this graphical interpretation of the three-dimensional stress state in terms of σ_n and τ, we note that the denominators of the above equations are positive also that $n_i^2 > 0$, all of which tells us that

$$(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\tau)^2 \geq 0$$

For the case where the equality sign holds, this equation may be rewritten, after some simple algebraic manipulations, to read

$$\left[\sigma_N - \frac{1}{2}(\sigma_{II} + \sigma_{III})\right]^2 + \tau^2 = \left[\frac{1}{2}(\sigma_{II} - \sigma_{III})\right]^2$$

Which is the equation of a circle in the σ_n, τ plane, with its center at the point on the σ_n axis, and having a radius $\frac{1}{2}(\sigma_{II} - \sigma_{III})$. We label this circle C1 and display it in Fig.40.

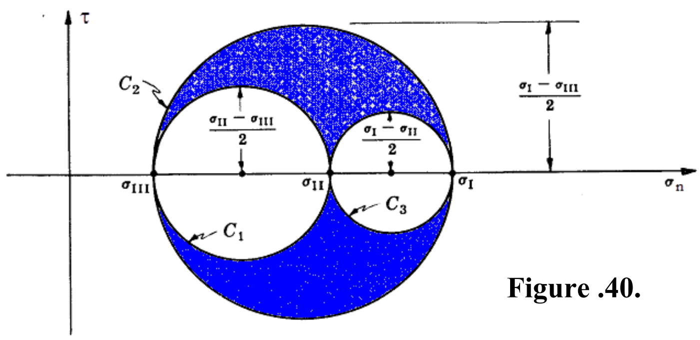

Figure .40.

We observe that conjugate pairs of values of σ_n and τ which satisfy this relationship result in stress points having coordinates exterior to circle C1.

Same work for the second and the third equations is done.
The three circles defined above, and shown in Fig.38, are called *Mohr's circles for stress*.

In order to relate a typical stress point having coordinates σ_n and τ in the stress plane of Fig. 40. to the orientation of the area element ΔS upon which the stress components σ_n and τ act, we consider a small spherical portion of the continuum body centered at P. As the unit normal n_i assumes all possible directions at P, the point of intersection of its line of action with the sphere will move over the surface of the sphere.

Accordingly, we may restrict our attention to the first octant of the spherical body, as shown in Fig.41. (Octant of small spherical portion of body together with plane at P with normal n_i referred to principal axes (O, x_1^*, x_2^*, x_3^*).

Let Q be the point of intersection of the line of action of n_i with the spherical surface ABC in Fig.41, and note that

$$\vec{n} = \cos\emptyset \,.\, \vec{e}_1^* + \cos\beta \,.\, \vec{e}_2^* + \cos\theta \,.\, \vec{e}_3^*$$

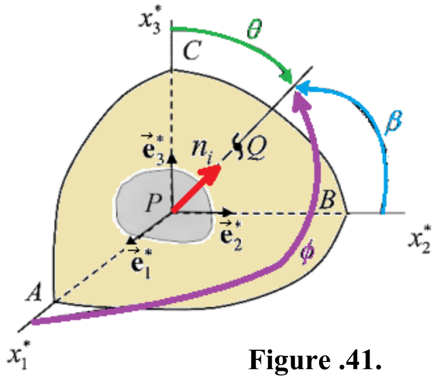

Figure .41.

The Mohr's stress semicircles for octant of small spherical portion are presented in Fig.42.

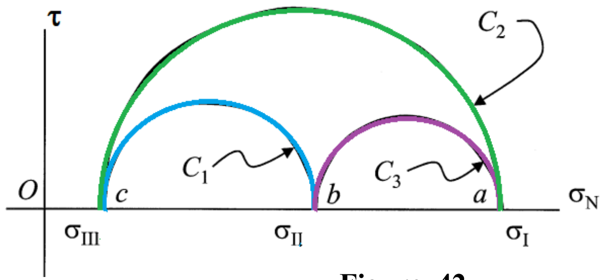

Figure .42.

The Fig.42 shows the reference angles φ and β for intersection point Q on surface of body octant.

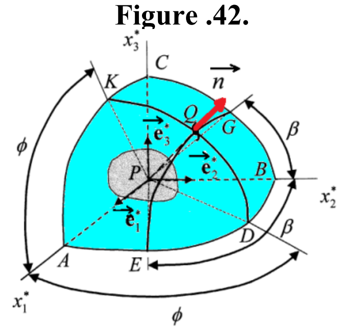

Figure .43.

The Mohr's semicircle for the precedent stress state displayed in Fig.43.

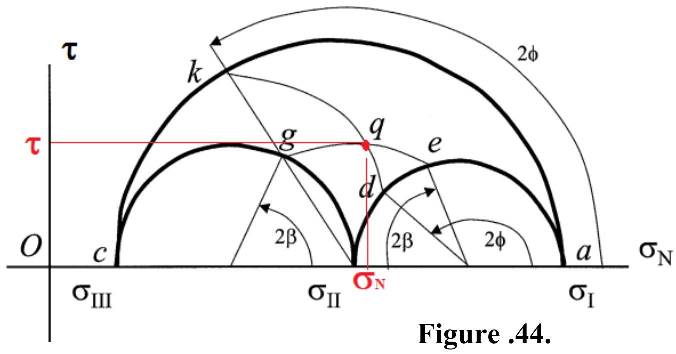

Figure .44.

In summary, for a specific at point P in the body, point Q, where the line of action of intersects the spherical octant of the body (Fig. 43.), is located at the common point of circle arcs KD and EG, and at the same time, the corresponding stress point q (having coordinates σN and τ) is located at the intersection of circle arcs kd and eg, in the stress plane of Fig.44. The following example provides details of the procedure.

Example:
The state of stress at point P is given in MPa with respect to axes $Px_1x_2x_3$ by the matrix:

$$\sigma_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix}$$

1/ determine the stress vector on the plane whose normal is $\vec{n} = \frac{1}{3}\left(2.\vec{e}_1 + \vec{e}_2 + 2.\vec{e}_3\right)$?

2/ determine the normal stress component σ_N and shear component τ on the same plane?

3/ Verify the results of part (2) by the Mohr's circle construction?

Solution:

1/ using the equation of the stress vector we find: $\overrightarrow{T^*}_{\vec{n}} = \sigma_{ij}.\vec{n}_j$

$$\begin{Bmatrix} T_{1\vec{n}} \\ T_{2\vec{n}} \\ T_{3\vec{n}} \end{Bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 50 \\ -150 \\ -50 \end{Bmatrix}$$

2/ the normal stress σ_N is determinate by :

$$\sigma_N = \begin{Bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{Bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{Bmatrix} = -\frac{150}{9} = -16.67\text{MPa} ;$$

The shear stress we use:

$$\tau^2 = \vec{T}_{\vec{n}}.\vec{T}_{\vec{n}} - \sigma_N^2 = \frac{1}{9}\left((50*50) + (150*150) + (50*50)\right) - (16.67)^2 = 2777;$$

Finally ; $\tau = 52.7$ MPa

3/ we can verify that for the stress tensor σ_{ij} given here the principal stress values are $\sigma_I = 50$, $\sigma_{II} = 25$ and $\sigma_{III} = -75$. Therefore, $\phi = \cos^{-1}\left(\frac{1}{3}\right) = 70.53^{o}$; $\theta = \beta = \cos^{-1}\left(\frac{2}{3}\right) = 48.19°$, so that — following the procedure outlined for construction of Fig.44. — we obtain Fig.45., from which we may measure the coordinates of the stress point q and confirm the values σ_N = –16.7 and τ = 52.7, both in MPa.

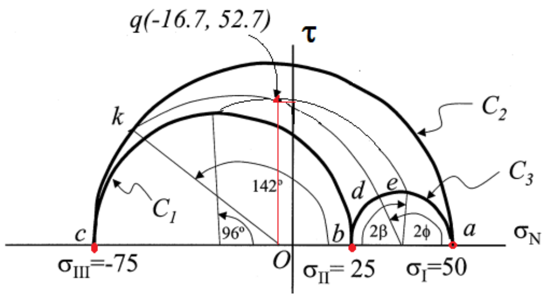

Figure .45.

Deviator and Spherical Stress State:

The arithmetic mean of the normal stresses,

$$\sigma_M = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii}$$

The state of stress having all three principal stresses equal (and therefore equal to σ_M) is called a *spherical state of stress*, represented by the diagonal matrix

$$\sigma_{ij} = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

for which all directions are principal directions.
Every state of stress σ_{ij} may be decomposed into a spherical portion S_{ij} and a portion D_{ij} known as the *deviator stress* in accordance with the equation

$$\sigma_{ij} = D_{ij} + \delta_{ij}\sigma_M = D_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$$

We notice immediately that the first invariant of the deviator stress is (the deviator diagonal is null)➔

$$D_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ii}\sigma_{kk} = 0$$

Example :
Decompose the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix}$$

into its deviator and spherical portions and determine the principal stress values of the deviator portion?

The arithmetic of mean normal stress is

$$\sigma_M = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(57 + 50 + 43) = 50$$

The spherical state of stress

$$S_{ij} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The *deviator stress*
$\sigma_{ij} = D_{ij} + S_{ij}$➔ $D_{ij} = \sigma_{ij} + \delta_{ij}\sigma_M$

$$D_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} - \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & -7 \end{bmatrix}$$

A convenient two-dimensional graphical representation of three-dimensional state of stress at a point is provided by the well-known *Mohr's stress circles*.

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