Matrices and Determinants

II.1. Introduction

In many applications analysis, variables are assumed to be related by sets of linear equations. Matrix of algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in convention of algebraic notation.

II.2. Matrix

A set of mn numbers (real or complex) arranged in rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called m×n matrix (read m by n matrix).

An m×n matrix is expressed as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{n} \\ a_{21} & a_{11} & \cdot\cdot & a_{11} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

Matrices are usually denoted by capital letter A,B,C and it's elements by small letters a,b,c…

1) **Order of a Matrix:**

(3,2) or (2,3), (m,n)

rows columns

$$\text{the} \quad \begin{bmatrix} 1 & 2 & 6 \\ 4 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} and \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 5 \\ 4 & 2 & 1 & 0 \\ 3 & 1 & 0 & 6 \end{bmatrix}$$

are matrices of orders (2×3) , (3×1) and (4×4) respectively.

2) **Null or Zero Matrix**

For example:

$$B= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a zero matrix of order } 2\times 4$$

The Matrix B_{mn} has the property that for every matrix A_{mn}.

$$A + B = B + A = A$$

3) Square Matrix

A Matrix A having same numbers of rows and column is called a square matrix.

The $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ are square matrices of order 2 and 3.

4) Diagonal matrix

The diagonal of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ consists of elements 1,5 and 9 in that order.

5) Scalar Matrix

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ are scalar matrices.

6) Identity Matrix or Unit Matrix

Thus $I_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the identity matrices of order 2 and 3.

7) Equal Matrices

$A = \begin{bmatrix} x+3 & y \\ z & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

A = B if it has a same order and $a_{ij} = b_{ij}$

By the definition of quality of matrices, we have:

$$A = B$$

$$\begin{cases} x + 3 = 2 \\ y = 5 \\ z = 3 \\ a = 4 \end{cases}$$

8) The Negative of Matrix

If $A_{3\times2} = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}$ then $-A_{3\times2} = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$

For every matrix $A_{m\times n}$, the matrix $-A_{m\times n}$ has the property that:

$$A + (-A) = (-A) + A = 0$$

II.3. Operation on Matrices

1) Multiplication of a matrix by a scalar

If $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix}$ then for a scalar k, $kA=\begin{bmatrix} k & 2k \\ 3k & 4k \\ 5k & 0 \end{bmatrix}$

For example: $5\begin{bmatrix} -1 & 2 & 0 \\ 3 & 5 & 2 \\ 1 & 0 & 4 \end{bmatrix}=\begin{bmatrix} -5 & 10 & 0 \\ 15 & 25 & 10 \\ 5 & 0 & 20 \end{bmatrix}$

2) Addition and subtraction of Matrices

A, B two matrices have a same order m×n ($A_{m\times n}$,$B_{m\times n}$)

$$A=\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \;,\; B=\begin{bmatrix} 4 & -5 \\ 3 & 2 \end{bmatrix}$$

$$A + B =\begin{bmatrix} 2+4 & 5+(-5) \\ 1+3 & 0+2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 4 & 2 \end{bmatrix}$$

$$A - B =\begin{bmatrix} 2-4 & 5-(-5) \\ 1-3 & 0-2 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ -2 & -2 \end{bmatrix}$$

Note:

1) A + B = B + A
2) (A + B) + C = A + (B + C)
3) The same or difference of two matrices of different order is not defined.

3) Product of matrices

Two matrices A and B with $A_{m\times p}$, $B_{p\times n}$ (p) the columns equal the rows.

The product of $A_{m\times p}$ and $B_{p\times n}$ is the matrix $(AB)_{m\times n}$

$$A=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \;,B= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A\times B=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1*2+2*1 & 1*1+2*1 \\ (-1*2)+3*1 & (-1*1)+3*1 \end{bmatrix}=\begin{bmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{bmatrix}=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$B\times A= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}=\begin{bmatrix} (2*1)+(1*-1) & (2*2)+(1*3) \\ (1*1)+(1*-1) & (1*2)+(1*3) \end{bmatrix}=\begin{bmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{bmatrix}=\begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}$$

Note:

- $A\times B \neq B\times A$

- $A \times A = A^2$, $A \times A^2 = A^3$, $A^2 \times A^2 = A^4$

Example $A= \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$, Find AB

Solution: $AB=\begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$

4) Determinants

- The determinant of a matrix is scalar (number)
- The determinant of a matrix is defined only for square matrix
- It is denoted by (det A) or $|A|$ for square matrix A

Example 01:

$$\mathbf{A}=\begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix} \text{, Find } |\mathbf{A}|.$$

$$A= \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix} = (3)\times(3)-(1)\times(-2)= 9+2=11$$

Example 02:

$$\mathbf{A}=\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix} \text{, Find } |\mathbf{A}|.$$

$$|\mathbf{A}|=\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix} = 3\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 0 & -2 \\ 1 & 4 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

$=3(4+6)-2(0+2)+1(0-1)$

$=30-4-1$

$=25$

$$|\mathbf{A}|=\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix} = 3\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - 0\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + 1\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$= 3(4+6)-0(6-3)+1(-4-1)$

$=30+(-5)$

$=25$

- If $|A|=0$, the A is singular.

II.4. Solution of linear equations by determinants

- Use Cramer's rule
- Use matrices

1) Transpose of a matrix

$$\text{If } A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \quad \text{then} \quad A^t=\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}$$

2) Symmetric Matrix:

A square matrix A is called symmetric if $A=A^t$

$$A=\begin{bmatrix}a & b & c\\ b & d & e\\ c & e & f\end{bmatrix}, \quad A^t=\begin{bmatrix}a & b & c\\ b & d & e\\ c & e & f\end{bmatrix} = A$$

Thus, A is symmetric.

3) Adjoint of a Matrix:

$$A=\begin{bmatrix}1 & 0 & -1\\ 1 & 3 & 1\\ 0 & 1 & 2\end{bmatrix}$$

Cofactor of A are:

$$C=\begin{bmatrix}5 & -2 & 1\\ -1 & 2 & -1\\ 3 & -2 & 3\end{bmatrix}, \quad C^{+}=\begin{bmatrix}5 & -1 & 3\\ -2 & 2 & -2\\ 1 & -1 & 3\end{bmatrix}$$

Hence $adjA=C^t$

Note: the adjoint of 2*2 matrix of $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is $adjA=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

4) Invers of a Matrix:

If A nonsingular square matrix, then $A^{-1} = \frac{adjA}{A}$

$$A=\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}$$

$$adjA=\begin{bmatrix}2 & -4\\ -1 & 3\end{bmatrix}, \qquad \mathbf{|A|=6-4=2}$$

hence $A^{-1}=\frac{1}{2}\begin{bmatrix}2 & -4\\ -1 & 3\end{bmatrix}$

Note:

$AB = BA= I$, $A_{n\times n}$,$B_{n\times n}$

$B= A^{-1}$ or $A=B^{-1}$

Example 01:

$$A=\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix},\ B=\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$$

$$AB= \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} =\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And

$$BA= \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence AB=BA=I

And therefore B= A^{-1} = $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example 02: find the inverse, if it exists, of the matrix

$$A=\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$$|A| =0+2(-2+3)-3(-2+3)=2-3= -1$$

$|A|$ = -1, so the solution exists.

Cofactor of A are

$$\begin{array}{lll} A_{11} = 0 & A_{12} = -1 & A_{13}= 1 \\ A_{21} = 2 & A_{22} = -3 & A_{23}= 2 \\ A_{31} = 3 & A_{32} = -3 & A_{33}= 2 \end{array}$$

Matrix of transpose of the cofactors is

$$adjA=C=\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

now, $A^{-1}=\frac{1}{|A|}\ adjA=\frac{1}{-1}\begin{bmatrix} 0 & 2 & 3 \\ 1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}=\begin{bmatrix} 1 & -2 & -3 \\ -1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

Example 03: Use matrices to find the solution set of

$$\begin{gathered} x+y-2z=3 \\ 3x-y+z=5 \\ 3x+3y-6z=9 \end{gathered}$$

Solution :

$$\text{Let } A=\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & 6 \end{bmatrix}$$

Since $|A|$ =3+21-24=0

Hence the solution of the given linear equations does not exist.

Example 04: use the matrices to find the solution set of

$$4x+8y+z=-6$$
$$2x-3y+2z=0$$
$$x+7y-3z=-8$$

$$\begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

$$\text{Let } A=\begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}, \quad |A| = -32+48+17= 61$$

Hence A^{-1} exists.

$$A\text{-} \quad \frac{1}{|A|}\, adjA=\frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since X= A^{-1}B, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}=\frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix}=\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

the solution set {(x,y,z)}={(-2,0,2)}.

Exercise 01

A. Find the inverse of matrix A:

$$A=\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{bmatrix} \end{array}$$

1) Det $= -4\begin{bmatrix} -1 & 3 \\ 4 & -3 \end{bmatrix}-(-6)\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}+2\begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}$

= -4(3-12+6(-15+6)+2(20-2)

$= -4(-9)+6(-9)+2(18)$

$= 36-54+36$

det=18

2) $adjA = ?$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = \begin{bmatrix} +\begin{bmatrix} -1 & 3 \\ 4 & -3 \end{bmatrix} & -\begin{bmatrix} 5 & 3 \\ -2 & -3 \end{bmatrix} & +\begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix} \\ -\begin{bmatrix} -6 & 2 \\ 4 & -3 \end{bmatrix} & +\begin{bmatrix} -4 & 2 \\ -2 & -3 \end{bmatrix} & -\begin{bmatrix} -4 & -6 \\ -2 & 4 \end{bmatrix} \\ +\begin{bmatrix} -6 & 2 \\ -1 & 3 \end{bmatrix} & -\begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix} & +\begin{bmatrix} -4 & -6 \\ 5 & -1 \end{bmatrix} \end{bmatrix} =$

$$\begin{bmatrix} -9 & 9 & 18 \\ -10 & -8 & 4 \\ -16 & 22 & 34 \end{bmatrix}$$

$$adjA = \begin{bmatrix} -9 & -10 & -16 \\ 9 & -8 & 22 \\ 18 & 4 & 34 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\, adjA = \frac{1}{18}\begin{bmatrix} -9 & -10 & -16 \\ 9 & -8 & 22 \\ 18 & 4 & 34 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{9} & -\frac{8}{9} \\ \frac{1}{2} & -\frac{4}{9} & \frac{11}{9} \\ \frac{1}{1} & \frac{2}{9} & \frac{17}{9} \end{bmatrix}$$

B. We must find the inverse of the matrix A

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\, adjA$$

$|A| = ?$

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -2 & -2 \end{bmatrix}$$

$$5 = -7 - (-12) = [(1) + (4) + (-12)] - [(-2) + (-4) + (-6)]$$

$|A| = 1$

adj A =?

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix} \\ \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix} \\ \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} (1)+4 & 2+4 & (-4)+2 \\ -1+6 & 1+6 & -2-2 \\ -2-3 & 2-6 & 1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 & -2 \\ -5 & 7 & +4 \\ -5 & +4 & 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 5 & -5 & -5 \\ -6 & 7 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 5 & -5 & -5 \\ -6 & 7 & 4 \\ +2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

C. Find the value of x given that:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$$

$$[1 \times 3][3 \times 3] = [1 \times 3]$$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1+4+1 & 2+0+0 & 0+2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = [0 + 4 + 4x] = 4 + 4x = 0$$

$$[1 \times 3][3 \times 1] = [1 \times 1]$$

$$4x = -4, \qquad x = -1$$

Exercise 02

1) Solving a system of equation with matrix inverse method, the number of variables must equal the number of equations

both equal 3 here

$$x - y + 3z = 2$$
$$2x + y + 2z = 2$$
$$-x - 2y + z = 3$$

By the matrix inverse method in three distinct steps, as follows:

1- Write the given system (above) as a single matrix equation

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$A \quad . \quad \mathrm{X} = \mathrm{B}$$

2- Solve the matrix equation obtained in step [1] above, finding X
We solve for the matrix variable X by left multiplying both sides of the above matrix equation ($AX = B$) By A^{-1}

$$\mathrm{A}^{-1} . \mathrm{A} . \mathrm{X} = \mathrm{A}^{-1} . \mathrm{B}$$

$$\mathrm{I} . \mathrm{X} = \mathrm{A}^{-1} . \mathrm{B}$$

$$\mathrm{X} = \mathrm{A}^{-1} . \mathrm{B} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{14}{5} \\ \frac{13}{5} \end{bmatrix}$$

3- Using step [2] above , write the solution to the original system

Original system | solution to original system

$$\begin{aligned} x - y + 3z &= 2 \\ 2x + y + 2z &= 2 \\ -x - 2y + z &= 3 \end{aligned} \quad \Rightarrow \quad \left\{ \begin{aligned} x &= -3 \\ y &= 2 \\ z &= \frac{13}{5} \end{aligned} \right.$$