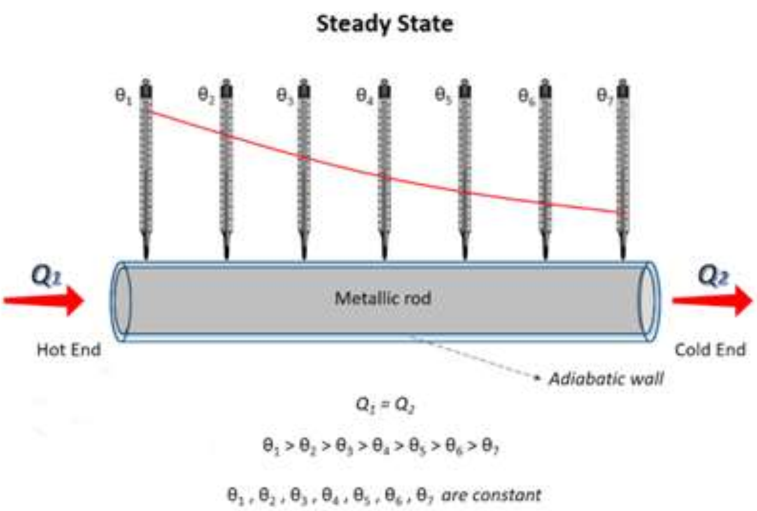
Chapter 2

Steady-state conduction heat transfer



Steady-state conduction heat transfer

2.1. Introduction

It is the transfer of heat within an opaque medium, without movement of material, under the influence of a temperature difference. The propagation of heat by conduction inside a body occurs according to two distinct mechanisms: transmission by the vibrations of atoms or molecules and transmission by free electrons.

2.2. Fourier's Law

A solid medium D and an elementary surface dS oriented by its unit normal n are considered in this case.

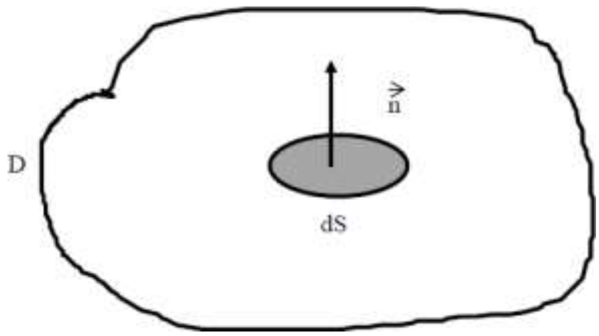


Figure 2. 1. Solid medium.

FOURIER's law determines the quantity of heat d2Q which passes through a surface dS in the direction of the normal n and over a time interval dt:

$$d^2Q = -\lambda \cdot \overrightarrow{gradT} \cdot \vec{dS} \cdot n \cdot dt \qquad (2.1)$$

With ;

$\overrightarrow{gradT}$: Temperature gradient calculated by:

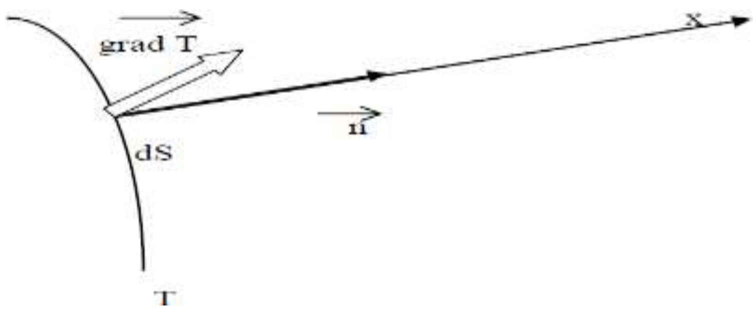
$$\overrightarrow{gradT}\begin{cases}\dfrac{\partial T}{\partial x}\\ \dfrac{\partial T}{\partial y}\\ \dfrac{\partial T}{\partial z}\end{cases}$$

λ: Thermal conductivity of the medium ($W.m^{-1}.K^{-1}$) We also have:

-The heat flow is: $$\boxed{d\Phi=\frac{d^2Q}{dt}=-\lambda\cdot\overrightarrow{gradT}\cdot dS\cdot\vec{n}} \quad (2.2)$$

-The heat flux density is: $$\boxed{d\varphi=\frac{d^2Q}{dt\cdot dS}=-\lambda\cdot\overrightarrow{gradT}\cdot\vec{n}} \quad (2.3)$$

The (-) sign indicates that in terms of heat flow, the gradient from largest to smallest is negative. If the surface dS is on an isothermal surface, the vectors *gradT* and *n* are collinear, as follows:



$$d^2Q=-\lambda\cdot\frac{dT}{dx}\cdot dS\cdot dt \quad (2.5)$$

Ou $d\Phi=-\lambda\cdot\dfrac{dT}{dx}\cdot dS$ et $d\varphi=-\lambda\cdot\dfrac{dT}{dx}$

2.3. Thermal conductivity

The following table represents the usual thermal conductivity values.

Table 2.1: thermal conductivity of different materials *(Values in W/m°C)*.

Substance Type	Thermal Conductivity (λ in W/m°C)
Gases at atmospheric pressure	0.006 - 0.15
Solid insulating materials *(Glass wool, polystyrene, cork, asbestos...)*	0.025 - 0.18
Non-metallic liquids	0.075 - 0.60
Non-metallic materials *(Brick, stone, wood, concrete...)*	0.10 - 2.2
Liquid metals	7.5 - 67
Metal alloys	12 - 100
Pure metals	45 - 365

2.4. Heat equation

2.4.1 In Cartesian coordinates

In the one-dimensional case, this equation expresses the unidirectional heat transfer through a plane wall:

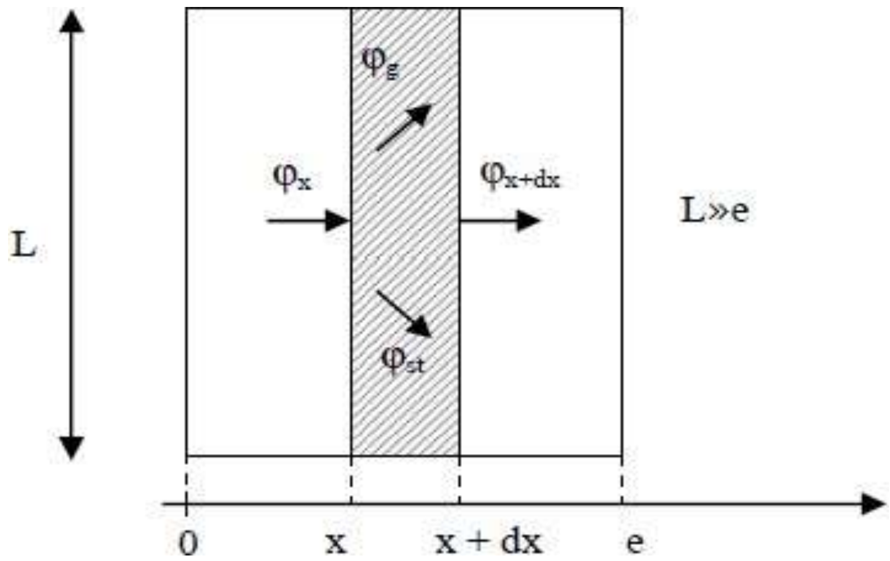


Figure 2. 2. Energy balance for an elementary system

The energy balance on this system is written:

$$\Phi_x + \Phi_g = \Phi_{x+dx} + \Phi_{st} \tag{2.6}$$

Avec :

$$\Phi_x = -\left(\lambda S \frac{\partial T}{\partial x}\right)_x \quad \text{et} \quad \Phi_{x+dx} = -\left(\lambda S \frac{\partial T}{\partial x}\right)_{x+dx}$$

$$\Phi_g = \dot{q}.dV = \dot{q}.S.dx$$

$$\Phi_{st} = \frac{\partial Q}{\partial t} = m.c.\frac{\partial T}{\partial t} = \rho.V.c.\frac{\partial T}{\partial t} = \rho.c.S.dx.\frac{\partial T}{\partial t}$$

By reporting in the energy balance and dividing by dx, we obtain:

$$\frac{\left(\lambda S \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda S \frac{\partial T}{\partial x}\right)_x}{dx} + \dot{q}.S = \rho.c.S.\frac{\partial T}{\partial t} \tag{2.7}$$

From Taylor's development:

$$\left(\lambda S \frac{\partial T}{\partial x}\right)_{x+dx} = \left(\lambda S \frac{\partial T}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\lambda S \frac{\partial T}{\partial x}\right)dx$$

Soit ;

$$\frac{\partial}{\partial x}\left(\lambda S \frac{\partial T}{\partial x}\right) + \dot{q}.S = \rho.c.S.\frac{\partial T}{\partial t} \tag{2.8}$$

In the three-dimensional (3D) case, we obtain the following thermal equation:

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + \overset{*}{q} = \rho.c.\frac{\partial T}{\partial t} \tag{2.9}$$

- **Study of particular cases:**

a) If the medium is isotropic and homogeneous (The conductivity only depends on the temperature of the point considered):

The heat equation (2.9) can be put in the form:

$$\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\partial \lambda}{\partial T}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] + \dot{q} = \rho.c.\frac{\partial T}{\partial t} \tag{2.10}$$

b) The temperature or presents a negligible difference:

This is particularly important in the case of a homogeneous and isotropic material with a constant λ coefficient. Consequently, the previous sentence becomes:

$$\boxed{\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=\rho.c.\frac{\partial T}{\partial t} \qquad (2.11)}$$

c) λ does not depend on the temperature and the internal heat release is negligible:

We have:

$$\boxed{\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\rho.c.\frac{\partial T}{\partial t} \qquad (2.12)}$$

Expression that we usually put in the form (called Poisson equation):

$$\boxed{a\nabla^2 T=\frac{\partial T}{\partial t}} \qquad (2.13)$$

The ratio $a = \lambda/_{\rho C}$ is called thermal diffusivity (m2.s-1) which characterizes the speed of propagation of a heat flow through a material.

d) Constant temperature (does not depend on time):

It is the analysis of a steady state with or without heat release. If we assume that the conductivity λ is a constant and does not depend on temperature, we obtain:

- ✓ if the internal source of heat: q ≠ 0

$$\boxed{\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\dot{q}=0 \qquad (2.14)}$$

- ✓ without internal heat release: q = 0, (we obtain the **Laplace** equation):

$$\nabla^2 T = 0$$

2.4.2. In cylindrical Cartesian coordinates:

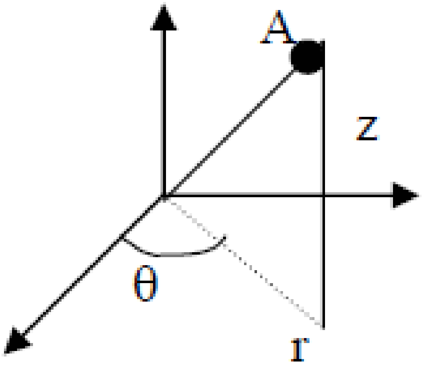


Figure 2. 3. Cylindrical coordinate system

Either:

$$\begin{bmatrix} x = r.\cos\phi \\ y = r.\sin\phi \\ z = z \end{bmatrix}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\lambda} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad (2.16)$$

If the case of a cylindrical symmetry problem where the temperature is only determined by r and t, equation (2.16) can be written as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{\lambda} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad (2.17)$$

2.4.3. In spherical coordinates:

Soient : $\begin{bmatrix} x = r.\cos\phi.\sin\theta \\ y = r.\sin\phi.\sin\theta \\ z = r.\cos\theta \end{bmatrix}$

$$\frac{1}{r}\left(\frac{\partial^2 (rT)}{\partial r^2}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial\varphi^2}\right)+\frac{\dot{q}}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t} \qquad (2.18)$$

2.5. Unidirectional conductive transfer:

2.5.1. Case of a simple wall

We assume that heat transfer is unidirectional and we neglect energy generation and storage.

Considering a homogeneous wall of thickness e, section S, thermal conductivity λ, whose faces are at temperatures T1 and T2, the heat flux which passes through this wall is:

$$\boxed{\Phi = -\lambda.S.\frac{dT}{dx}} \qquad (2.19)$$

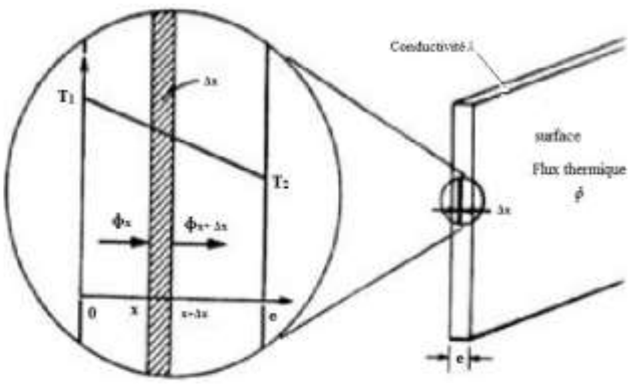


Figure 2. 4. Single wall

If we carry out an energy balance on the system (S) which contains a section of wall between the abscissa x and x +dx, it comes:

$$\boxed{\Phi_x = \Phi_{x+dx} \Rightarrow -\lambda S\left(\frac{\partial T}{\partial x}\right)_x = -\lambda S\left(\frac{\partial T}{\partial x}\right)_{x+dx}} \qquad (2.20)$$

From where ;

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = 0}$$

With the boundary conditions: T (x = 0) = T1 and T (x = e) = T2

D'où
$$T = T_1 - \frac{x}{e}(T_1 - T_2) \qquad (2.21)$$

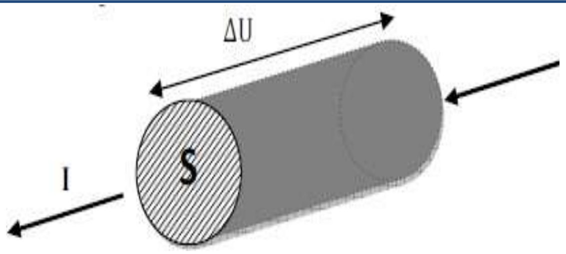
We find a linear profile for the temperature. The heat flow passing through the wall is calculated by:
$$\boxed{\Phi = -\lambda.S.\frac{dT}{dx}}$$

$$\boxed{\Phi = -\lambda.S.\frac{T_2 - T_1}{e} = \lambda.S.\frac{T_1 - T_2}{e} \qquad (2.22)}$$

- Concept of thermal resistance (electrical analogy):

An electric wire of length L, section S and thermal conductivity Ke subjected to a potential difference ΔU, lets an electric current pass such that:

$$\boxed{\Delta U = U_0 - U_1 = \frac{L}{K_e \cdot S} \cdot I \text{ ou encore ; } \Delta U = R_e \cdot I}$$



This equation is analogous to that giving the thermal potential:

$$\Delta T = T_1 - T_2 = \left(e\ \lambda \cdot S\right) \cdot \Phi = R_{th} \cdot \Phi \tag{2.23}$$

We establish the following correspondence:

Table 2.2: Electrical-thermal analogy.

Thermal System 🔥	Analogy ⇔	Electrical System ⚡
🔷 Fourier's Law	⇔	🔷 Ohm's Law
$\Delta T = -\left(\frac{e}{\lambda S}\right) \cdot \Phi$	⇔	$\Delta U = RI$
🔶 Thermal Conductivity (λ)	⇔	🟦 Electrical Conductivity (σ)
$\lambda(T)$	⇔	$\sigma(T)$
🌡 Temperature (T)	⇔	⚡ Electric Potential (V)
🔥 Thermal Power (Φ)	⇔	🔋 Current Intensity (I)
🛑 Thermal Resistance (R_th)	⇔	⬜ Electrical Resistance (R)
$R_{th} = \frac{e}{\lambda S}$	⇔	R

For the thermal problem, we establish its electrical equivalent:

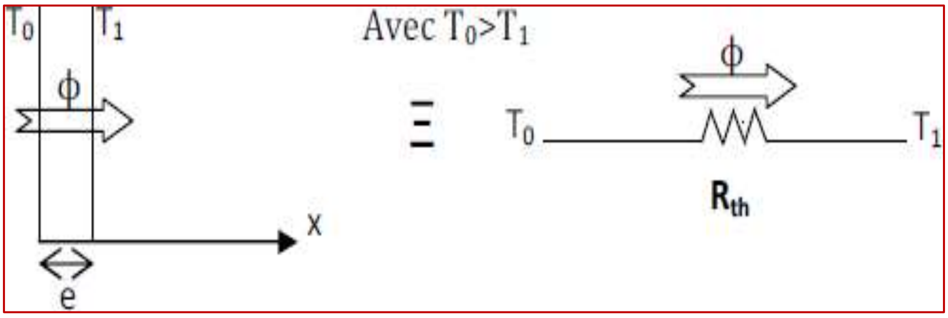


Figure 2. 5. Equivalent electrical diagram of a simple wall

In the case of a flat wall of thickness e, the thermal resistance is:

$$\boxed{R_{th} = \frac{e}{\lambda . S} \quad \text{en °C/W.}} \tag{2.24}$$

The total thermal resistance is determined as in electricity, we know that:

1) Two thermal resistances are added in the case where they are in series:

$$R_{eq} = R_1 + R_2$$

2) The inverse of the thermal resistances are added in the case where they are in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

2.5.2. Case of a multi-layer wall

In steady state, the heat flow is conserved when crossing the wall and is written:

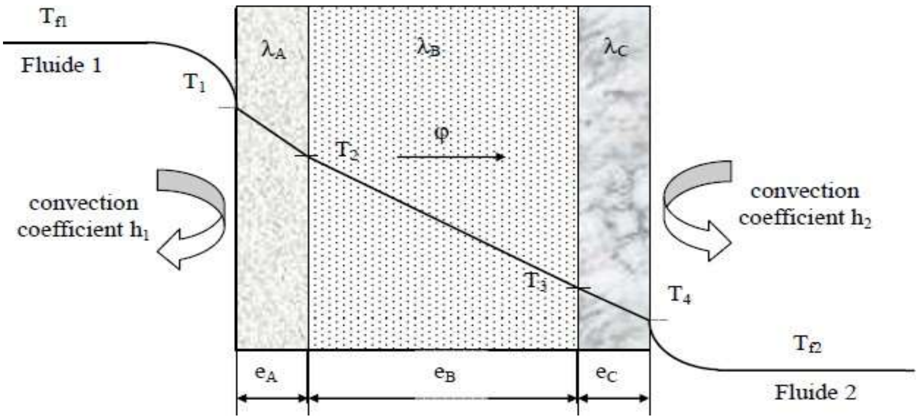


Figure 2. 6. multi-layer wall

$$\Phi = h_1 S\left(T_{f1} - T_1\right) = \frac{\lambda_A S\left(T_1 - T_2\right)}{e_A} = \frac{\lambda_B S\left(T_2 - T_3\right)}{e_B} = \frac{\lambda_C S\left(T_3 - T_4\right)}{e_C} = h_2 S\left(T_4 - T_{f2}\right)$$

$$\Phi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + \frac{e_B}{\lambda_B S} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S}} \qquad (2.25)$$

The thermal resistance of a multi-layer wall is:

$$R_{th} = \frac{T_{f1} - T_{f2}}{\Phi} = \frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + \frac{e_B}{\lambda_B S} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S} \qquad (2.26)$$

The interactions between the different types of layers were considered perfect and there was no temperature discontinuity at the interfaces. Indeed, due to the roughness of the surfaces, a microlayer of air exists between the hollows of the surfaces in view, which contributes to the formation of thermal resistance (air is an insulator) called thermal contact resistance. The previous equation then becomes:

$$\Phi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{h_A S} + R_{AB} + \frac{e_B}{h_B S} + R_{BC} + \frac{e_C}{h_C S} + \frac{1}{h_2 S}} \qquad (2.27)$$

The equivalent electrical diagram is shown in Figure 2.7.

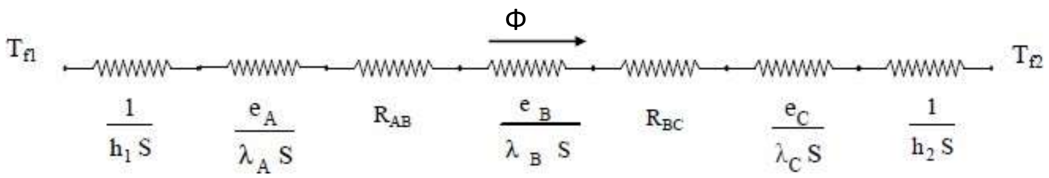


Figure 2. 7. Equivalent electrical diagram of a multi-layer wall.

2.5.3. Case of a composite wall:

The equivalent thermal resistance R of a portion of wall of width L and height using the laws of association of resistances in series and in parallel by the relationship (figure 2.8):

$$R_{th} = R_1 + R_2 + \frac{1}{\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}} + R_6 + R_7 \qquad (2.28)$$

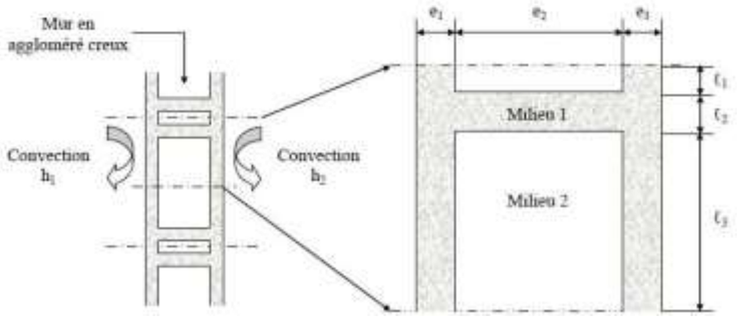


Figure 2. 8. Composite wall

With:

$$R_1 = \frac{1}{h_1 l L} \; ; \; R_2 = \frac{e_1}{\lambda_1 l L} \; ; \; R_3 = \frac{e_2}{\lambda_2 l_1 L} \; ; \; R_4 = \frac{e_2}{\lambda_1 l_2 L} \; ; \; R_5 = \frac{e_2}{\lambda_2 l_3 L} \; ; \; R_6 = \frac{e_3}{\lambda_1 l L} \; ; \; R_7 = \frac{1}{h_2 l L}$$

The equivalent electrical diagram is given in Figure 2.9.

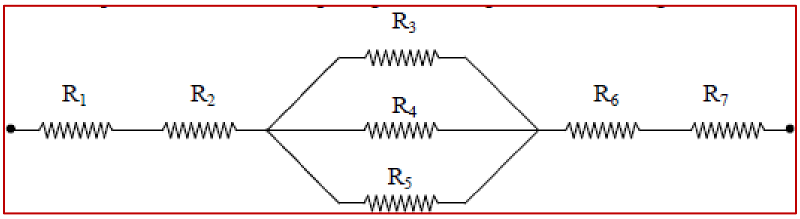


Figure 2. 9. Equivalent electrical diagram of a composite wall

2.5.4. Case of a long hollow cylinder (tube)

We assume a hollow cylinder with thermal conductivity ʎ interior and exterior of the cylinder, T_1 and T_2 are the temperatures of the internal and external walls (figure 2.10). The temperature gradient is considered to vary only radially.

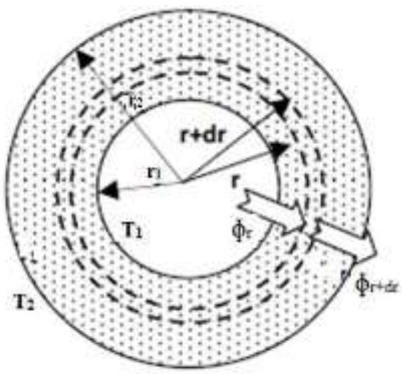


Figure 2. 10. Diagram of transfers in a hollow cylinder

Let us carry out the thermal balance of the system constituted by the part of the cylinder between the radii r and r + dr:

$$\Phi_r = \Phi_{r+dr} \tag{2.29}$$

Avec : $\Phi_r = -\lambda 2\pi r L\left(\frac{dT}{dr}\right)_r$ et $\Phi_{r+dr} = -\lambda 2\pi(r+dr)L\left(\frac{dT}{dr}\right)_{r+dr}$

Soit : $-\lambda 2\pi r L\left(\frac{dT}{dr}\right)_r = -\lambda 2\pi(r+dr)L\left(\frac{dT}{dr}\right)_{r+dr}$ d'où $r\frac{dT}{dr} = C$

Avec les conditions aux limites : $T(r_1) = T_1$ et $T(r_2) = T_2$

D'où :

$$\frac{T(r)-T_1}{T_2-T_1} = \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \tag{2.30}$$

And by application of the relation $\Phi = -\lambda 2\pi r L\left(\frac{dT}{dr}\right)_r$ we obtain: This relation can also be put in the form:

$$\boxed{\Phi = \frac{2\pi\lambda L(T_1-T_2)}{\ln\left(\frac{r_2}{r_1}\right)} \qquad (2.31)}$$

$$\boxed{\Phi = \frac{(T_1-T_2)}{R_{12}} \quad \text{avec} \quad R_{12} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda L}}$$

This relationship can also be put in the form:

And be represented by the equivalent electrical diagram in Figure 2.11.

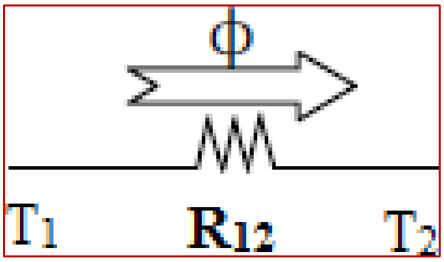


Figure 2. 11. Equivalent electrical diagram of a hollow cylinder

2.5.5. Case of a multilayer hollow cylinder

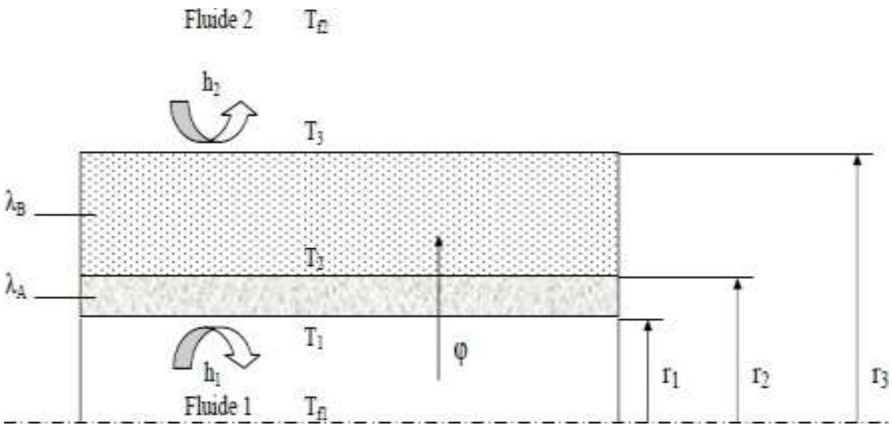


Figure 2. 12. Diagram of transfers in a multilayer hollow cylinder.

In steady state, the heat flow φ is preserved when crossing the different layers and is written (figure 2.12):

$$\Phi = h_1 2\pi r_1 L\left(T_{f1} - T_1\right) = \frac{2\pi\lambda_A L\left(T_1 - T_2\right)}{\ln\left(\frac{r_2}{r_1}\right)} = \frac{2\pi\lambda_B L\left(T_2 - T_3\right)}{\ln\left(\frac{r_3}{r_2}\right)} = h_2 2\pi r_3 L\left(T_3 - T_{f2}\right) \tag{2.32}$$

D'où :

$$\Phi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 2\pi r_1 L} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda_A L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi\lambda_B L} + \frac{1}{h_2 2\pi r_3 L}} \tag{2.33}$$

Which can be represented by the equivalent electrical diagram in Figure 2.13.

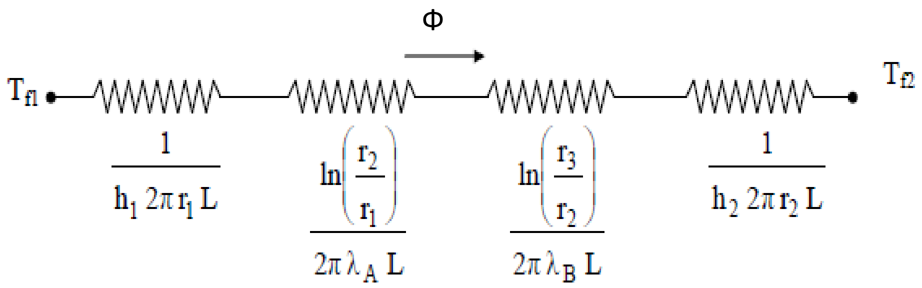


Figure 2. 13. Equivalent electrical diagram of a multilayer hollow cylinder.

- The thermal resistance of a multilayer hollow cylinder is:

$$\boxed{R_{th}=\frac{T_{f1}-T_{f2}}{\Phi}=\frac{1}{h_1 2\pi r_1 L}+\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda_A L}+\frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi\lambda_B L}+\frac{1}{h_2 2\pi r_3 L}}$$

2.5.7 Fin theory

2.5.7.1 General

The fins are good conductors of heat with a greater surface area than the rest of the body. They are used to improve heat removal from a solid-state containment system with high flux densities.

Finned tube (radiator)

Heat sinks

Motorcycle engine.

Figure 2. 14. Examples of finned systems used in different application sectors

2.5.7.2. The bar equation

Performing the energy balance, the temperature profile in the bar is the solution of the following differential equation, called the bar equation:

$$\boxed{\frac{d^2T}{dx^2}-\frac{h.P_e}{\lambda.S}.(T-T_\infty)=0}$$

2.5.7.2.1. Flow extracted by a fin

A fin is a medium that is a good conductor of heat, one dimension of which is large compared to the others. By posing:

$$\boxed{\omega^2=\frac{h.P_e}{\lambda\,S}\quad et\ \theta=(T-T_\infty)}$$

The previous equation becomes:

$$\boxed{\frac{d^2\theta}{dx^2}-\omega^2.\ \theta=0}$$

2.6. Multi-directional conductive transfer

In the case where the heat diffusion does not take place in a single direction, two resolution methods can be applied:

2.6.1. Form coefficient method

In two-dimensional or three-dimensional systems where only two limiting temperatures T_1 and T_2 intervene, we show that the heat flow can be put in the form:

$$\boxed{\varphi=\lambda.F(T_1-T_2)}$$

With :

ʎ : Thermal conductivity of the medium separating the surfaces S1 and S2 (W m^{-1} K^{-1})

T_1: Surface temperature S_1 (K)

T_2: Surface temperature S_2 (K)

F: Shape coefficient (m)

The shape coefficient F only depends on the shape, dimensions and relative position of the two surfaces S_1 and S_2.

2.6.2. Numerical methods

In the case where the shape coefficient method cannot be applied (non-isothermal surfaces for example), the heat equation must be solved numerically:

- Finite difference method
- Finite element method.
- Finite volume method…