Chapter 4

Frequency responses of linear systems

4.1 Introduction

The study of frequency response concerns the study of the response of the system when it is subjected to sinusoidal inputs of different frequencies. The frequency response can be viewed in two ways: through the Bode diagram or Nyquist diagram.

4.2 Frequency response (harmonic) linear systems

Rather than performing a mathematical decomposition of the input and output signals into elementary sinusoidal functions, it appeared simpler to apply a harmonic input to the system input and to record the corresponding output, once the steady state has been established. This operation is repeated starting from the lowest frequencies up to frequencies high enough for the system output to be negligible.

In fact, from a certain frequency the system no longer "follows" the oscillations that we would like to impose on it, exactly like the eardrum which no longer vibrates beyond a certain frequency.

4.3 Bode diagram

The Bode diagram allows us to study the frequency response of a linear system with a transfer function $G(p)$. To do this, we replace p by $j\omega$, which allows the transfer function to be written in the following form:

$$G(j\omega) = A(\omega)e^{j\varphi(\omega)} \tag{4.1}$$

In this expression, we consider the module $A(\omega)$ and the argument $\varphi(\omega)$ of the complex function parameterized by the pulsation ω. The Bode plot is obtained by plotting (asymptotically) the following functions of $A(\omega)$ And $\varphi(\omega)$ on logarithmic scales on the abscissa ω (Fig.4.1). The Bode diagram consists of plotting the open-loop transfer function $T(j\omega)$, a gain curve for the module and a phase curve for the argument.

$$A_{db} = 20\, log_{10} \left| T(j\omega) \right| \\ \varphi = \arg\left(T(\omega)\right) \tag{4.2}$$

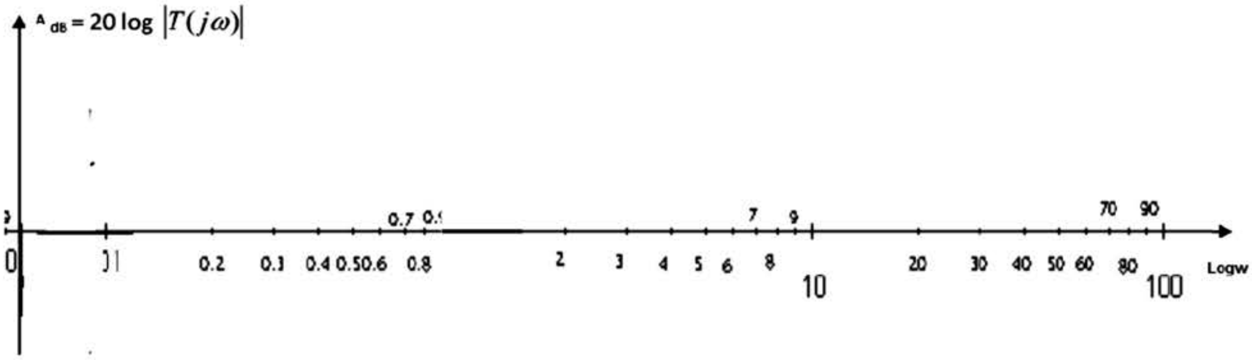

Fig.4.1: Example of a three-decade semi-logarithmic scale.

Remarks:

1) Term: $T(p) = K.p^{\alpha}$

$T(j\omega) = K.(j\omega)^{\alpha}$

$A_{db} = 20\log_{10}\left|T(j\omega)\right| = 20\log_{10}\left|K.(j\omega)^{\alpha}\right|$

$A_{db} = 20\log_{10}\left(K\omega\right)^{\alpha} = 20\log_{10} K + 20.\alpha.\log_{10}\omega$

$A_{db} = a\log_{10}\left(\omega\right) + b$

$\varphi = arctg(\omega)$

$A_{db} = ax + b$: a straight line.

- We plot the function A_{db} depending on $\log$, the decibel (dB), it is the unit of gain.
- To draw the line, you will need to choose the slope in decades or octaves.

Decade it is the difference between in a logarithmic scale 10ω And ω.

$A_{db/déc} = A(10\omega) - A(\omega) = 20.\log_{10} K + 20.\alpha.\log_{10}(10\omega) - \left[20.\log_{10} K + 20.\alpha.\log_{10}(\omega)\right]$

$A_{db/déc} = 20.\alpha.\log_{10}(10\omega) - 20.\alpha.\log_{10}(\omega)$

$A_{db/déc} = 20.\alpha.\log_{10}\dfrac{10\omega}{\omega} = 20.\alpha \quad db/déc$

Octave it is the difference between in a logarithmic scale 2ω And ω.

$A_{db/oct} = A(2\omega) - A(\omega) = 20.\log_{10} K + 20.\alpha.\log_{10}(2\omega) - [20.\log_{10} K + 20.\alpha.\log_{10}(\omega)]$

$A_{db/oct} = 20.\alpha.\log_{10}(2\omega) - 20.\alpha.\log_{10}(\omega)$

$A_{db/oct} = 20.\alpha.\log_{10} \dfrac{2\omega}{\omega} = 6.\alpha \quad db/oct$

$A_{db} = 0 \Rightarrow K\omega_c^{\alpha} = 1 \Rightarrow \omega_c^{\alpha} = \dfrac{1}{K} \Rightarrow \omega_c = \left(\dfrac{1}{K}\right)^{1/\alpha}$: Is called the cutoff frequency where the line intersects the axis of y.

1) Term: $T(p) = (1+\tau p)^{\beta}$

We will study this function with $\beta = 1$ And $\beta = -1$

$\beta = 1$, Let us consider a system with a transfer function:

$T(p) = (1+\tau p) \Rightarrow T(j\omega) = (1+\tau j\omega)$

$|T(j\omega)| = \sqrt{1+\tau^2\omega^2}$

$\varphi = arctg(\tau\omega)$

$A_{db} = 20\log_{10}|T(j\omega_c)| = 0 \Rightarrow \omega_c = \dfrac{1}{\tau}$

Note that one of the advantages of the Bode plot is that the overall diagram is the algebraic sum of the partial diagrams. Some examples of plots of Bode curves will be given.

Example 4.1:

We consider an open-loop transfer function system:

$$G(p) = p \qquad (4.2)$$

To study the frequency response of this system, we replace p by $j\omega$,To use the Bode representation, we look for the expressions for the gain and phase of the transfer function.

4.3.1 Bode diagram of a first-order system

We consider the transfer function system:

$$G(p) = \frac{1}{1+\tau p} \qquad (4.6)$$

τ is a positive constant (the time constant)

The asymptotic diagram is obtained by the following reasoning. We study the gain at low frequencies and high frequencies. The low and high frequencies are determined relative to a cut-off frequency ω_c which is defined by: $\omega_c = \dfrac{1}{\tau}$.

- If $\omega << \omega_c$, the low frequency gain is: $G_{db} = -10\log_{10}(1) = 0\ dB$. So in low frequencies: $\omega = 0$ until ω_c the decibel gain is zero.
- If $\omega >> \omega_c$, the high frequency gain is: $G_{db} = -20\log_{10}(\tau\omega)$. So in high frequencies $\omega = \omega_c$ until $\omega = \infty$, the gain is asymptotically linear.

Likewise, for the phase shift:

- If $\omega << \omega_c$, the phase shift at low frequencies is: $\varphi = -arctg(0) = 0°$ .So in low frequencies: $\omega = 0$ until $\omega = \omega_c$,the decibel shift is practically zero.
- If $\omega >> \omega_c$, the phase shift at high frequencies is: $\varphi = -arctg(\infty) = -90°$. So in high frequencies $\omega = \omega_c$ until $\omega = \infty$, the phase shift is asymptotically equal to 90°.

Having determined the asymptotic behavior, it remains to specify the values of the gain and the phase shift in the vicinity of the cut-off frequency. This is obtained by setting $\omega = \omega_c$.

$$G_{db}(\omega = \omega_c) = -10\log_{10}(2) = -3dB \tag{4.10}$$

For the phase shift at the cutoff frequency, we have:

$$\varphi = -arctg(\tau\omega_c) = -arctg(1) = -45° \tag{4.11}$$

Example 4.2:

Plot the Bode diagram of the transfer function of a first-order system with a static gain equal to 1 and a time constant equal to $2\mu s$:

$$H(p) = \frac{1}{1+\tau p} = \frac{1}{1+2.10^{-6}p}$$

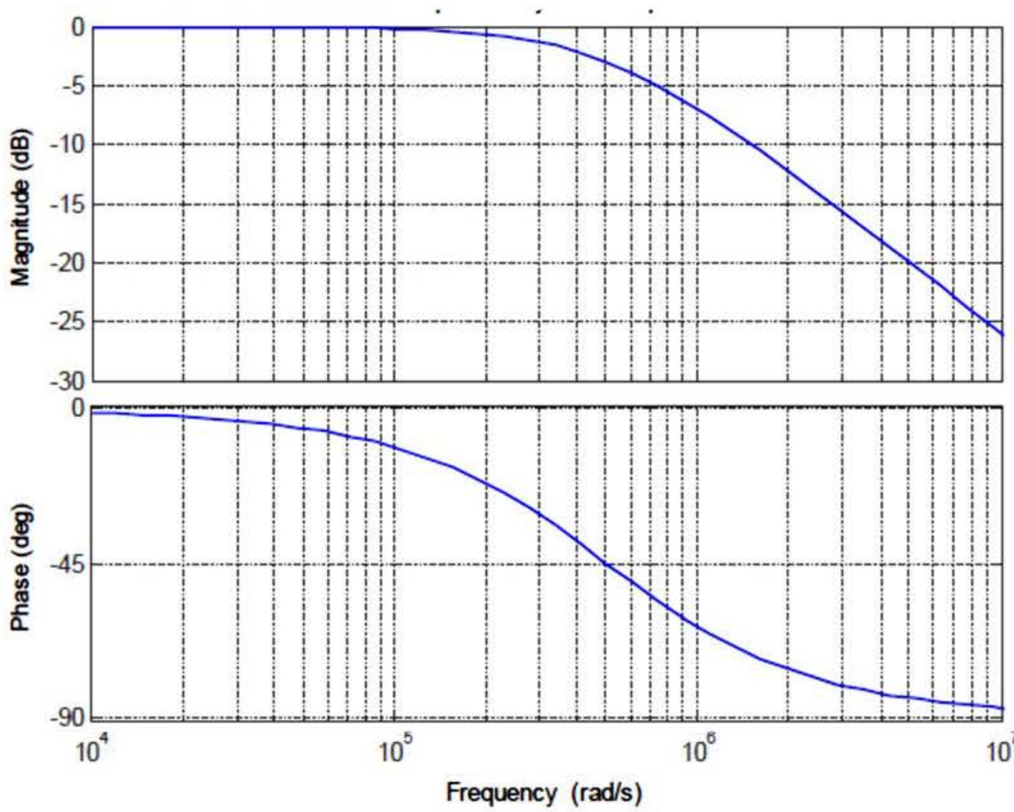

Fig.4.4: Bode locus of a first-order system.

4.3.2 Bode diagram of a 2nd order system

Let the transfer function system be:

$$G(p)=\frac{\omega_n^2}{p^2+2\xi\omega_n p+\omega_n^2},\quad 0<\xi<1 \qquad (4.12)$$

We have already determined the expressions for gain and phase shift which are:

$$\begin{aligned}|G(j\omega)|&=\frac{1}{\sqrt{\left(1-\omega_n^2\right)^2+4\xi^2\omega_n^2}}\\ \varphi&=-\arctan\left(\frac{2\xi\omega_n}{\left(1-\omega_n^2\right)}\right)\end{aligned} \qquad (4.13)$$

<u>Asymptotic behavior</u>

The expression for the gain is:

$$G_{db} = -20\log_{10}\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}} \tag{4.14}$$

- In low frequencies, $\frac{\omega}{\omega_n} << 1$, the gain is: $G_{db} \cong -20\log_{10}(1) \approx 0\ dB$. The asymptote at low frequencies is zero.
- In high frequencies, $\frac{\omega}{\omega_n} >> 1$, the gain is: $G_{db} \approx -20\log_{10}\left(\frac{\omega}{\omega_n}\right)$ The high frequency asymptote is a slope of -40 dB/decade.
- If the input signal pulse is ω_c, SO $\frac{\omega_c}{\omega_n} = 1$, the gain is: $G_{db} = -20\log_{10}\left[(2\xi)^2\right]^{\frac{1}{2}}$

This gain for the cutoff frequency depends on the damping coefficient.
If we assume $\xi = 1$, the gain is $-6dB$.
A similar reasoning is carried out to determine the asymptotic diagram of the phase shift.

$$\varphi = -\arctan\left(\frac{2\xi\omega_n}{(1-\omega_n^2)}\right) \tag{4.15}$$

- In low frequencies, $\frac{\omega}{\omega_n} << 1$, the gain is: $-\varphi \cong 0°$. The asymptote at low frequencies is zero.
- In high frequencies, $\frac{\omega}{\omega_n} >> 1$, the gain is $\varphi = -\arctan\left(\frac{2\xi\omega_n}{(1-\omega_n^2)}\right) \approx -\arctan(0) = -180°$.

  The high frequency asymptote is a horizontal line at $-180°$.
- If the input signal pulse is ω_c, SO $\frac{\omega_c}{\omega_n} = 1$, the phase shift is: $\varphi(\omega_c) = -\arctan(\infty) = -90°$.

In Figure 4.5, the Bode diagram (amplitude and phase) is given for:

$$G(p) = \frac{1}{p^2 + 0,5p + 1}, \tag{4.16}$$

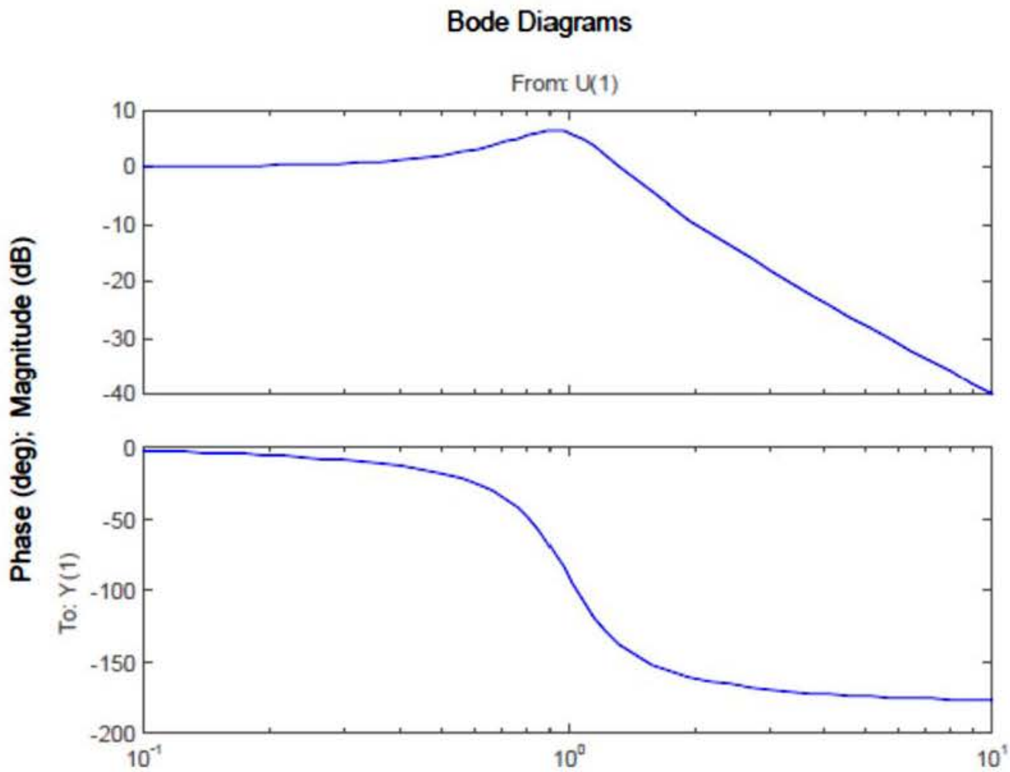

Fig.4.5: Asymptotic Bode diagram of a 2nd order system.

The resonance pulsation is the one that cuts the axis$(0dB)$ at the pulsation: $\omega = \omega_r$, with : r the resonance.

Deriving equation (4.14) for $\omega = \omega_r$:

$$\frac{dG_{dB}}{d\omega} = 0 \Rightarrow \omega_r = \omega_n\sqrt{1-2\xi^2} \text{ if } 2\xi^2 < 1 \text{ or else } \xi < 0,7$$

- If ξ very small: $\omega_r = \omega_n$ and Case of G_{dB} max $= \dfrac{1}{1-\dfrac{\omega_r^2}{\omega_n^2} + j2\xi\dfrac{\omega_r}{\omega_n}}$
- If $\xi < 0.7$: the module has passed through a Max G_{dB} max $= \dfrac{1}{2\xi\sqrt{1-\xi^2}}$. This maximum is obtained by: $\omega = \omega_r = \omega_n\sqrt{1-\xi^2}$.

The resonance factor or surge factor: $Q = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$.

4.3.3 Phase margin

We call it phase margin, noted M_φ , the deviation between the -180° value and the phase of the zero gain point pulse point ω_{0dB} :

$$M_\varphi = Arg\left(T\left(j\omega_{0dB}\right)\right) + 180° \quad (4.17)$$

The system is stable in closed loop if, for this pulsation, the corresponding phase is strictly greater than -180°.
It is common to want a phase margin of 45°.

4.3.4 Profit margin

We call it the profit margin, noted M_G , the gap between the pulse point gain $\omega_{-180°}$ the phase point pulsation equal to -180° and the value $0\ dB$:

$$M_G = -20\log_{10}\left|T\left(j\omega_{-180°}\right)\right| \quad (4.18)$$

The system is closed-loop stable if, for this pulsation, the corresponding gain is strictly negative.
It is common to want a profit margin of $10\ dB$.

4.3.5 Graphical interpretations

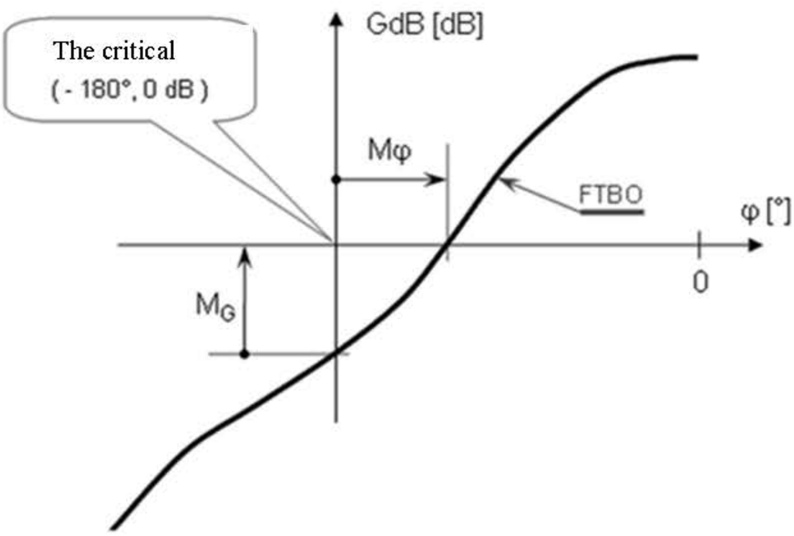

Fig.4.6: Phase Margin and Gain Margin of Bode Plot.

4.4 Nyquist Location

Gain and phase curves are not the only possible representation of the harmonic behavior of a transfer function system. $G(p)$. There is another representation which consists of decomposing the transfer function into its real and imaginary part.

$G(p)$ relative to an open loop system, the Nyquist diagram is the locus of the points defined:

- **In polar coordinates**: Defined by a radius vector equal to the value of the modulus of $G(j\omega)$ and a polar angle equal to the argument of $G(j\omega)$. The location is graduated relative to the frequency. ω .

$$G(j\omega) = |G(j\omega)|e^{j\varphi(\omega)} = |G(j\omega)|(\cos\varphi + j\sin\varphi) \tag{4.19}$$

- **In rectangular coordinates**: Defined by a curve giving the variation of the imaginary part as a function of the real part of the transfer function.

$$G(\omega) = \mathrm{Re}(\omega) + j\,\mathrm{Im}(\omega) \tag{4.20}$$

This formulation is used by Nyquist to represent the harmonic response of a given system by its transfer function, it is the plot of the polar coordinate curve $G(\omega)$ And $\varphi(\omega)$ when ω varies from 0 to +∞. This curve is called the Nyquist locus or polar locus. (Figure 4.7)

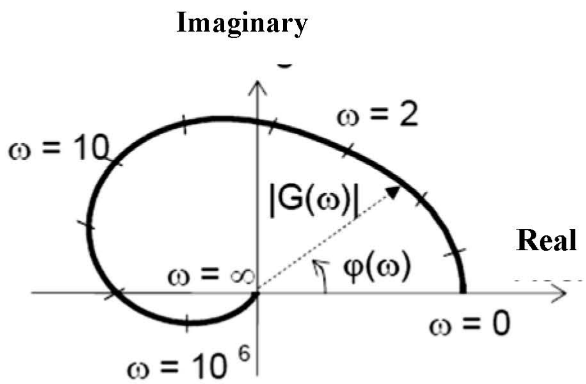

Fig.4.7: Nyquist curve.

We generally complete this curve by symmetry with respect to the real axis to have its representation when ω varies from 0 to – ∞. This part of the curve has no physical meaning, but it is necessary to plot it if we want to study mathematically the stability of the control.

We can thus complete the curve because the coefficients of the differential equation which governs the phenomenon are assumed to be constant and real.

Like Bode plots, the Nyquist locus can be plotted either experimentally or from the transfer function if it is known, which is why it is also called the "transfer locus in the Nyquist plane".

4.4.1 Nyquist locus of a first-order system

Let us consider a first-order system of transfer function: $G(p)=\dfrac{K}{1+Tp}$

With T : the time constant. On the other hand, if we separate the real and imaginary parts of the transfer function, we have:

$$G(j\omega)=\frac{K}{1+jT\omega}=\frac{K\left(1-jT\omega\right)}{1+T^2\omega^2}\Rightarrow \mathrm{Re}(\omega)=\frac{K}{1+T^2\omega^2},\ \mathrm{Im}(\omega)=\frac{-KT\omega}{1+T^2\omega^2} \tag{4.21}$$

It is easy to find the different points of this curve (Fig.4.8):

$$\omega\to 0\begin{cases}\mathrm{Re}\to K\\ \mathrm{Im}\to 0\end{cases},\quad \omega\to\infty\begin{cases}\mathrm{Re}\to 0\\ \mathrm{Im}\to 0\end{cases},\quad \omega\to 1\begin{cases}\mathrm{Re}\to \dfrac{K}{1+T^2}\\ \mathrm{Im}\to \dfrac{-KT}{1+T^2}\end{cases} \tag{4.22}$$

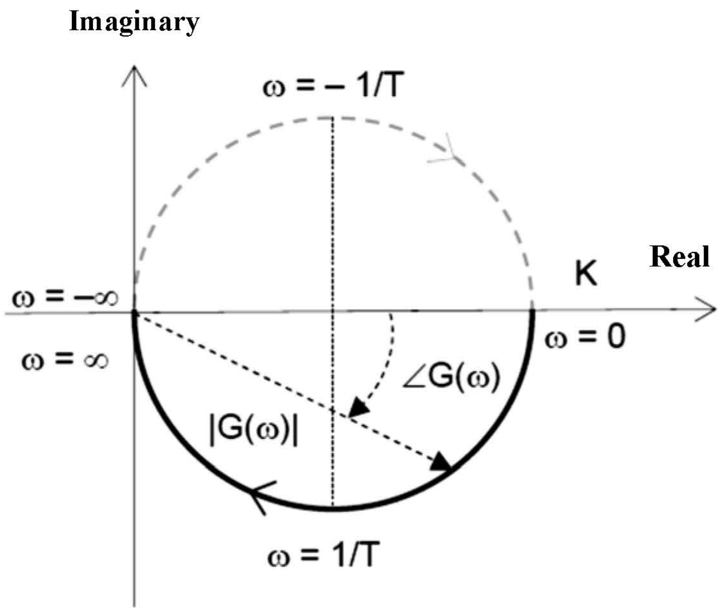

Fig.4.8: Nyquist diagram of a first-order system.

Example 4.3:

Let the transfer function system be: $G(p)=\dfrac{1}{1+p}$.

To establish the Nyquist locus, we replace p by $j\omega$ In $G(p)$ and we break down $G(j\omega)$ into a real part and an imaginary part.

4.4.2 Nyquist locus of a 2nd order system

The transfer function of a 2nd order system: $G(p)=\dfrac{K\omega_n^2}{p^2+2\xi\omega_n p+\omega_n^2}$.

This time we trace $G(j\omega)=|G(j\omega)|e^{j\varphi(\omega)}$ in polar coordinates.

The shape of the curve (Fig. 4.10) can be deduced from the Bode curves. We see in particular that:

$$\omega=0\Rightarrow\begin{cases}|G(\omega)|=K\\ \varphi=0°\end{cases}\quad,\quad \omega\to\infty\begin{cases}|G(\omega)|=0\\ \varphi=-180°\end{cases}\tag{4.23}$$

It is difficult to equate the curve because it does not have a known classical shape.

- For a system with very little or no damping, resonance occurs for ω, so for $tg\varphi=-\infty$, either $\varphi=-90°$.
- The larger ξ, the "smaller" the curve is and the closer it is to that of a first-order system (semi-circle).
- When $\omega\to\infty$, $\varphi\to-180°$. The curves are therefore tangent to the real axis.

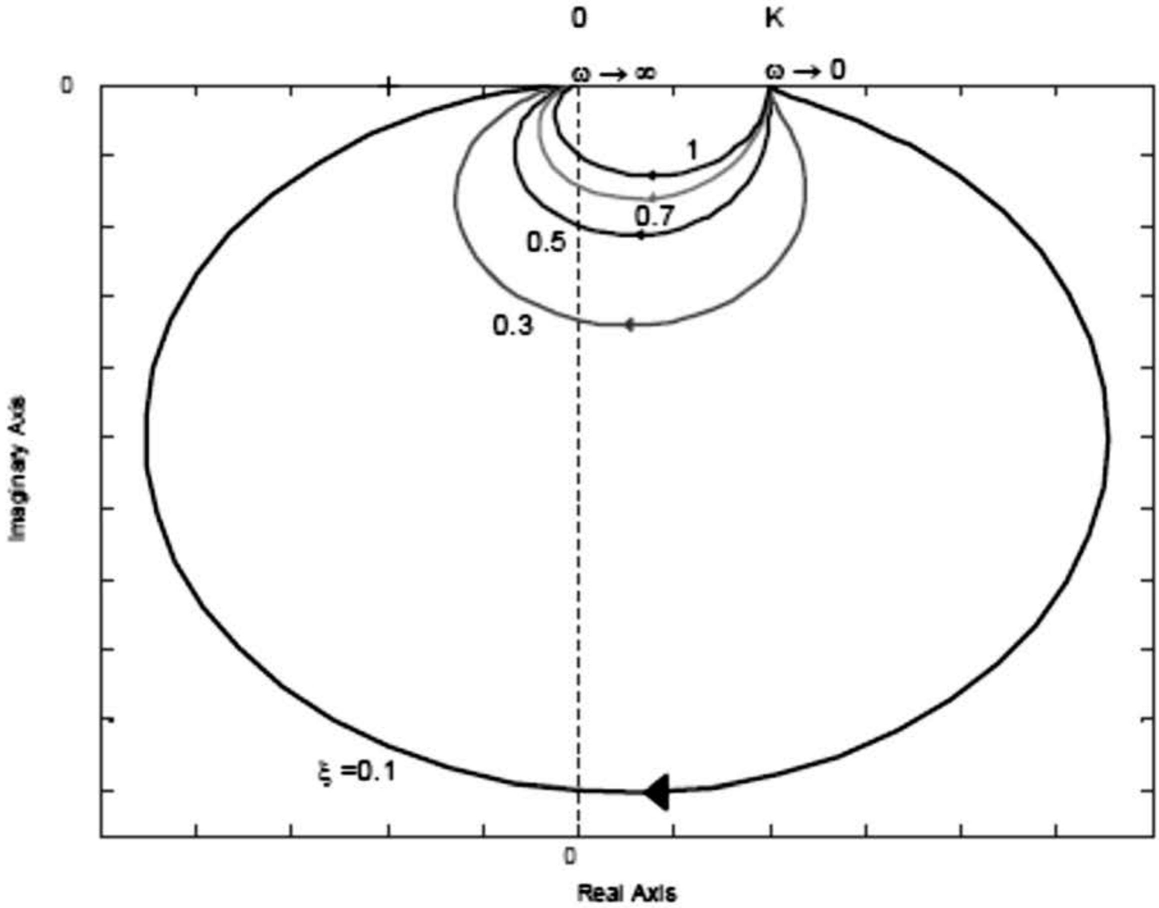

Fig.4.10: Nyquist locus of a 2nd order system.

Example 4.4:

Let the FT system be:

$$H(p) = \frac{1}{(1+p)(2+p)}$$

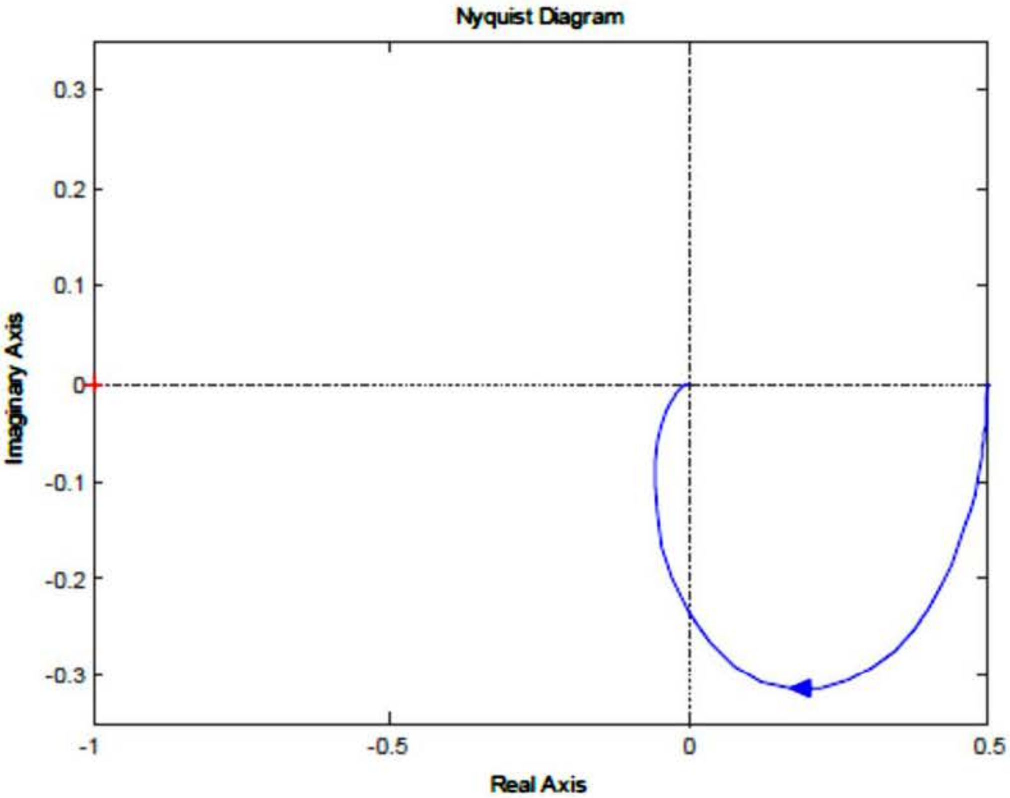

Fig.4.11: Nyquist locus of the FT $G(p)=\dfrac{1}{(1+p)(2+p)}$ **by MATLAB.**