Chapter 3

Time responses of linear systems

3.1 Introduction

We want to characterize systems on the one hand by their transfer function and, on the other hand, by their behavior. The latter can be highlighted by its response to a given input.

3.2 Time response of linear systems

Typically, one can learn a lot about systems by observing the response to the following inputs:

- The impulse: impulse response.
- The level: index response
- The ramp: response to a ramp
- The sinusoid: frequency response

In the first part, we will make the link between transfer function and time responses (i.e. impulse, step and ramp responses). In the rest of the course, we will study simple and widespread systems that are first-order and second-order systems. In addition, the methods for studying these systems are easily generalized to others.

3.3 Study of a first-order servo system

A first-order system is a system governed by a first-order linear differential equation such as:

$$\tau \frac{ds(t)}{dt} + s(t) = Ke(t) \tag{3.1}$$

The two constants τ And K are positive real numbers, in general.

τ : is called the time constant of the system.

K :is called static gain.

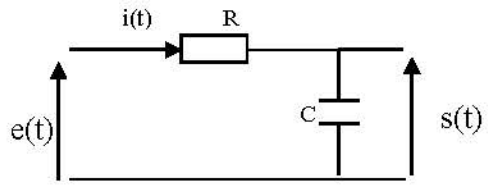
These systems are sometimes called single time constant systems.

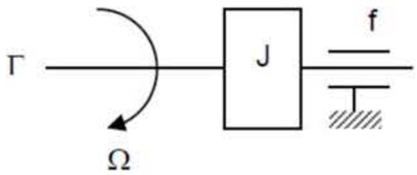
Taking the initial conditions as zero, the transfer function of the system is calculated from the differential equation using the Laplace transform:

$$\tau p S(p) + S(p) = K.E(p) \Rightarrow S(p)(\tau p + 1) = K.E(p)$$

$$\frac{S(p)}{E(p)} = \frac{K}{\tau p + 1} \tag{3.2}$$

3.3.1 Examples





3.3.2 Response of a first-order system to some canonical signals

For the time study, we seek the answers $s(t)$ systems studied at signals entry $e(t)$ called canonical entries, which are generally:

- Dirac's impulse $\delta(t)$, (impulse response)
- the unit level $u(t)$, (step response)
- unit ramp $r(t)$.

A. Response to a Dirac impulse

Let be a first-order linear system. The input to the system is a Dirac impulse $e(t) = \delta(t)$ its Laplace transform is $E(p) = 1$.

So the impulse response of a first order system is:

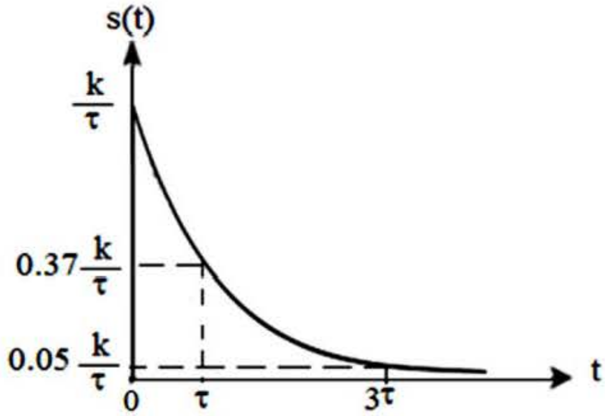


Fig.3.1: Impulse response of a first-order system.

t	0	τ	3τ	$+\infty$
$s(t)$	$\frac{K}{\tau}$	$0{,}37\frac{K}{\tau}$	$0{,}05\frac{K}{\tau}$	0

B. Response to a step

The input consists of an amplitude step E_0 $e(t) = E_0 u(t)$.

So the step response of a first order system is:

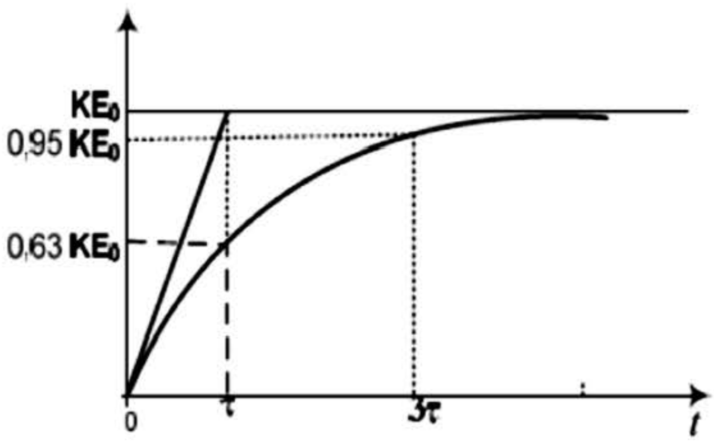


Fig.3.2: Step response of a first-order system

t	0	τ	3τ	4τ	$+\infty$
$s(t)$	0	$0{,}63KE_0$	$0{,}95KE_0$	$0{,}98KE_0$	KE_0

For all step responses (at one level) we define:

- Steady state:

$$s_p(t) = s(t) \;\; \forall t >> t_r \;\; (s_p(t) = \lim_{t\to\infty} s(t))$$

- Rise time t_m :

  This is the time it takes for the response to evolve from 10 to 90%, from 5 to 95%, or from 0 to 100% of its final value.

  For poorly damped 2nd order systems, the rise time from 0 to 100% is more commonly taken.

  For highly damped systems, the 10 to 90% evolution is more often adopted.

- Response time at 5% t_r :

  Is the time at the end of which: $\forall t > t_r \quad s_p(t) - s(t) < 0.05 s_p(t)$

- The overtaking:

  Either $S_{\max}$ the maximum value taken by $s(t)$ And S_f is the final value (reached in steady state). The maximum excess is defined if $S_{\max} > S_f$ by :

$$D = S_{\max} - S_f .$$

  The relative excess is a percentage according to: $D\% = \left(\frac{S_{\max} - S_f}{S_f} \times 100 \right) \%$

- Temporal characteristics:

  The transient characteristics of a system do not depend on the amplitude of the applied step but only on the time constant τ of this system.

System	$\frac{S(p)}{E(p)} = \frac{K}{\tau p + 1}$
Exceeding $D\%$	0
Rise time t_m	2.2τ
Response time to $5\% t_r$	3τ

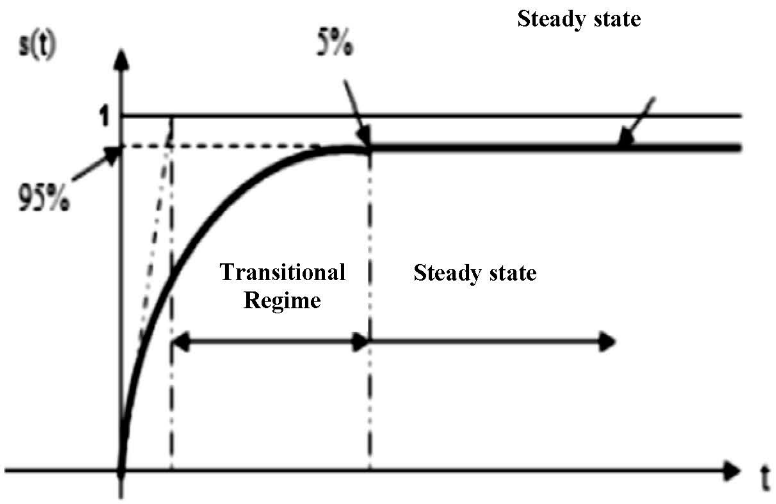


Fig.3.3: The temporal characteristics of a first-order system with a unit step input.

C. Response to a ramp

Let us consider a first-order linear system. The input to the system is a ramp with slope a_0 , $e(t) = r(t) = B.t.u(t)$, and its Laplace transform is $E(p) = \frac{B}{p^2}$.

$$\frac{S(p)}{E(p)} = \frac{K}{\tau p + 1} \Rightarrow S(p) = \frac{K}{\tau p + 1} E(p)$$

$$E(p) = \frac{B}{p^2} \Rightarrow S(p) = \frac{-\tau BK}{p} + \frac{B}{p^2} + \frac{\tau BK}{p + \frac{1}{\tau}}$$

So the response of a 1st order system to a ramp input is:

$$s(t) = BK(t - \tau + \tau e^{-\frac{t}{\tau}}) \tag{3.5}$$

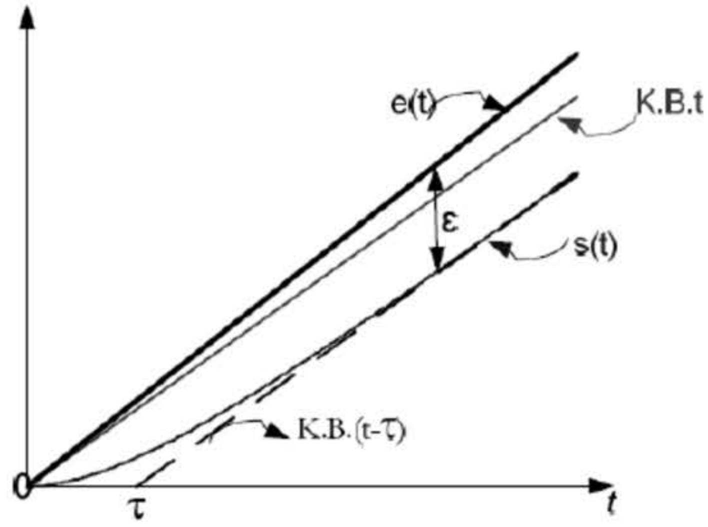


Fig.3.4: Response to a ramp of a 1st Order system.

3.4 Study of a second-order servo system

The most general second-order differential equation is:

$$b_2 \frac{d^2 s(t)}{dt^2} + b_1 \frac{ds(t)}{dt} + b_0 s(t) = a_2 \frac{d^2 e(t)}{dt^2} + a_1 \frac{de(t)}{dt} + a_0 e(t) \qquad (3.6)$$

In this paragraph we will only study systems such that the input derivatives do not intervene $(a_2 = a_1 = 0)$. The transfer function of these systems can be put in the canonical form:

$$H(p) = \frac{K}{1 + \frac{2\xi}{\omega_n} p + \frac{1}{\omega_n^2} p^2} \qquad (3.7)$$

With:

K : The static gain of the system.

ω_n : The natural pulsation (in rad/s).

ξ : The damping coefficient.

If we look for the poles of the transfer function:

$$1 + \frac{2\xi}{\omega_n} p + \frac{1}{\omega_n^2} p^2 = 0$$

With : $\sqrt{\Delta'} = \omega_n \sqrt{\xi^2 - 1}$

There are three possible cases:

- $\xi > 1$ In this case the poles are real: $-\xi\omega_n \mp \omega_n\sqrt{\xi^2 - 1}$.
- $\xi = 1$ The two poles are equal and real. They are worth $-\omega_n$.
- $\xi < 1$ The two poles are complex conjugates. They have negative real parts if $\xi > 0$.

3.4.2 System Responses to Different Input Signals

A. Response to a unit impulse

Let be a second order linear system. The input to the system is a Dirac impulse $e(t) = \delta(t)$ and its Laplace transform is $E(p) = 1$:

$$E(p) = 1 \Rightarrow S(p) = \frac{K\omega_n^2}{p^2 + 2\xi\omega_n p + \omega_n^2} \tag{3.8}$$

- If $0 < \xi < 1 \Rightarrow s(t) = K\left(\frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n t\sqrt{1-\xi^2}\right)\right) \quad t \geq 0$
- If $\xi = 1 \Rightarrow s(t) = K\omega_n^2 t e^{-\omega_n t} \quad t \geq 1$
- If $\xi > 1 \Rightarrow s(t) = K\frac{\omega_n}{2\sqrt{\xi^2 - 1}}\left(e^{-\left(\xi-\sqrt{\xi^2-1}\right)\omega_n t} - e^{-\left(\xi+\sqrt{\xi^2-1}\right)\omega_n t}\right) \quad t \geq 1$

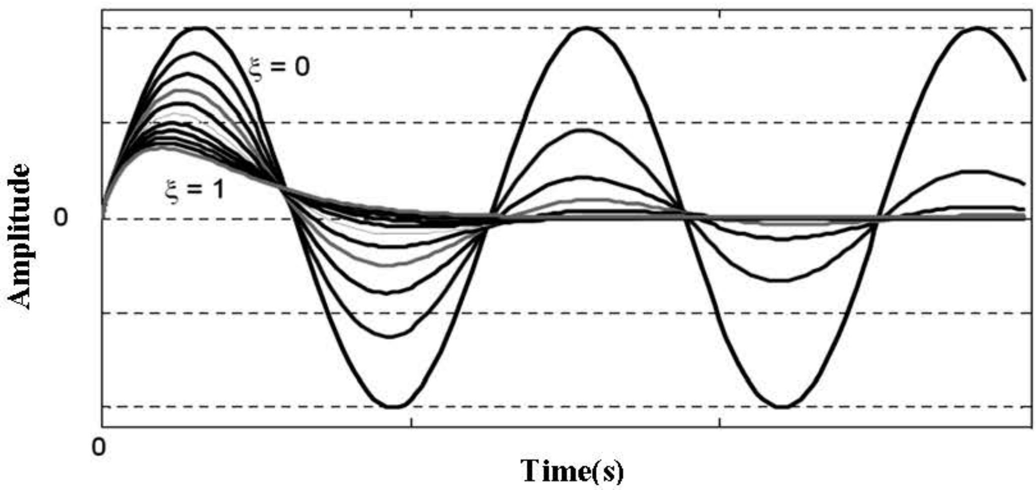


Fig.3.5: Impulse responses of a 2nd order system as a function of ξ.

B. Response at a unit level

The entry consists of a unit step $e(t) = u(t)$.

$$E(p) = \frac{1}{P} \Rightarrow S(p) = \frac{1}{P}\frac{K\omega_n^2}{p^2 + 2\xi\omega_n p + \omega_n^2}$$

If we take $K = 1$ and we discuss the system according to the values of the damping coefficient, we will have 3 cases to study.

- If $\xi < 1$: $s(t) = 1 + \frac{1}{2}\frac{\xi}{\sqrt{\xi^2 + 1}} e^{-\left(\xi - \sqrt{\xi^2+1}\right)\omega_n t}$
- If $\xi = 1$: $s(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)$ $\rightarrow$ The system is in critical damping.
- If $0 < \xi < 1$: $s(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t + \varphi\right)$

With : $\varphi = arctg\frac{\xi}{\sqrt{1-\xi^2}}$ $\rightarrow$ The system being in critical damping mode.

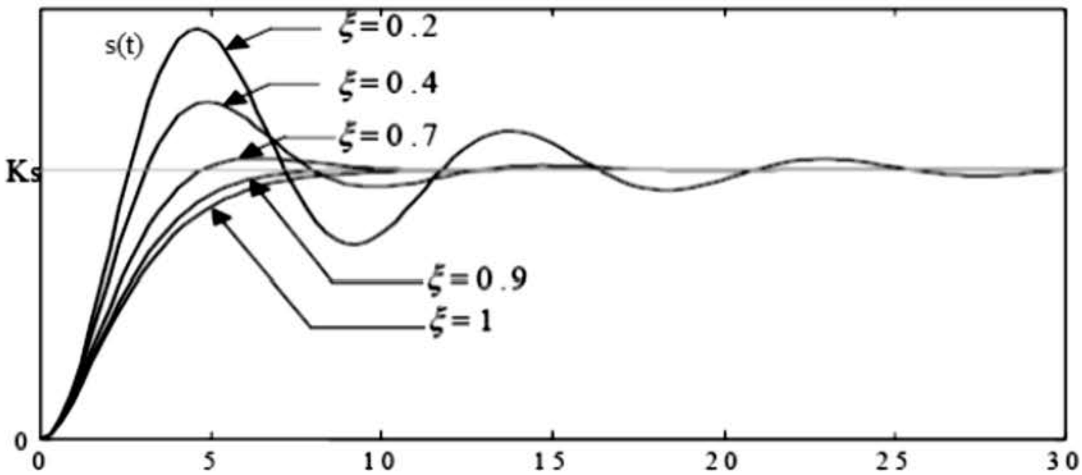


Fig.3.6: Step responses of a 2nd order system as a function of ξ .

Example 3.1:

Given a system written by the following functional diagram, find the variation domains of k for the three possible regimes.

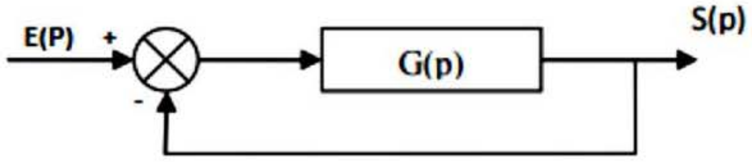


Fig.3.7: Unitary return system.

With : $G(p) = \dfrac{K}{p(p+2)}$

<u>Temporal characteristics</u>:

The transient characteristics of a system do not depend on the amplitude of the applied step but on two factors: ξ (damping coefficient) and ω_n (proper pulse) of this system.

System	$\dfrac{S(p)}{E(p)} = \dfrac{K\omega_n^2}{p^2 + 2\xi\omega_n p + \omega_n^2}$	
	$\xi < 1$	$\xi \geq 1$
Exceeding $D\%$	$100.e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}\%$	0
Peak time t_p	$\dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$	
Response time to $5\% t_r$	$-\dfrac{1}{\xi\omega_n}\ln\left(0.05\sqrt{1-\xi^2}\right)$	$\dfrac{3}{\omega_n\left(\xi - \sqrt{1-\xi^2}\right)}$

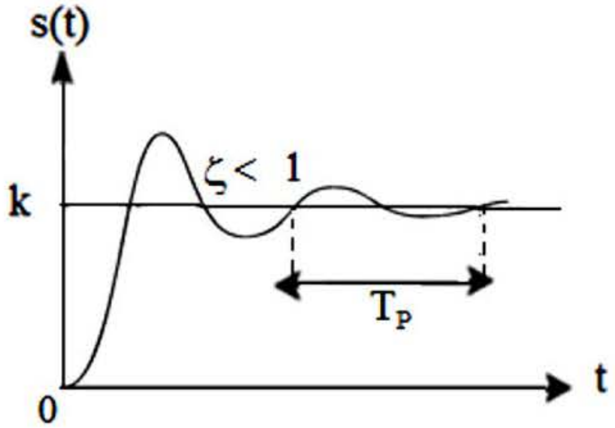


Fig.3.7: Step response of a second-order system with damping coefficient less than 1.

<u>**Remarks:**</u>

- The oscillations are all the more important (in amplitude and duration) as the damping factor is low. If it is zero, the output signal is a sinusoid oscillating between 0 and $2K$.
- Of the value of ξ damping factor, depends on the type of system response.
- Amortized regime : $\xi > 1$. In the case of the damped regime, the output of the system tends all the more slowly towards its final value. k that ξ is big.
- Critical regime : $\xi = 1$ The critical regime is characterized by the fastest possible response: the signal $s(t)$ tends very quickly towards its final value, without oscillations.
- Damped oscillatory regime : $\xi < 1$. In the case of the damped oscillatory regime, the pulsation of the sinusoidal signal enveloped by the decreasing exponential has the expression: $\omega_p = \omega_n\sqrt{1-\xi^2}$.

  This is the pseudo-pulsation of the damped oscillatory regime. It is lower than the pulsation ω_n .The pseudo-period of these oscillations is also defined by:

$$T_p = \frac{2\pi}{\omega_p} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

  This pseudo-period is equal to the time interval corresponding to a complete alternation of the damped sinusoid (see figure 3.7). It is all the greater as ξ is close to 1.

3.5 Harmonic Response

It is the response of a system to a periodic input such as sinusoidal functions. It allows the study of the system in steady state. This will be studied in chapter 4, with Bode and Nyquist diagrams.