PW 04: NUMERICAL INTEGRATION

I. INTRODUCTION

In this chapter we look at ways of calculating an approximation to a definite integral of a real-valued function of a single variable, $\boldsymbol{f(x)}$ (the **integrand**). Sometimes integration can be achieved using standard formulae, possibly after some manipulation to express the integral in a particular form. However, often standard formulae cannot be used (for example, the function may be known at a discrete set of points only) and so it is necessary to resort to numerical techniques.

Overview: All the methods that we look at involve approximating an integral with a weighted sum of evaluations of the integrand. The methods differ in some or all of the following aspects:

- the spacing of the points at which the integrand is evaluated.
- the weights associated with the function evaluations.
- the accuracy that a method achieves for a given number of function evaluations.
- the convergence characteristics of the method (how the approximation improves with the addition of new function values).

In particular, we look at:

- Central rectangle
- Trapezium method.
- Simpson's method.

II. CENTRAL RECTANGLE

The integral $I = \int_a^b f(x)dx$ is calculated as the area of a rectangle whose height is $f\left(\frac{a+b}{2}\right)$. The area of the rectangle is therefore $(b-a).f\left(\frac{a+b}{2}\right)$.

- This method is not widely used because of the large approximation error it causes.

III. TRAPEZIUM METHOD

Find $\boldsymbol{I = \int_a^b f(x)dx}$ by the Trapezium method

1 We approximate the integral $I = \int_{x0}^{x1} f(x)dx$ with $\int_{x0}^{x1} P_1(x)dx$ where $P_1(x)$ is the interpolation polynomial of $f(x)$ on the points $x0$ and $x1$.

2 Using the Lagrange method:

$$\boldsymbol{L_0(x)} = \frac{\boldsymbol{(x - x_1)}}{\boldsymbol{(x_0 - x_1)}}$$

$$\boldsymbol{L_1(x)} = \frac{\boldsymbol{(x - x_0)}}{\boldsymbol{(x_1 - x_0)}}$$

$$\boldsymbol{P_1(x) = y_0 L_0(x) + y_1 L_1(x)}$$

$$\boldsymbol{P_1(x) = y_0 \frac{(x - x_1)}{(x_0 - x_1)} + y_1 \frac{(x - x_0)}{(x_1 - x_0)}}$$

3 The generalized Trapezoid integral is given by:

$$\boldsymbol{I_{Trap}} = \int_{x0}^{xn} P_1(x)\boldsymbol{dx} = \boldsymbol{\frac{h}{2}[y_0 + y_n + 2\sum_{i=1}^{n-1} y_i]}$$

IV. SIMPSON'S METHOD

Find $\boldsymbol{I = \int_a^b f(x)dx}$ by the Simpson's method

1 For this method, we approximate the integral $I = \int_{x0}^{x2} f(x)dx$ with $\int_{x0}^{x2} P_2(x)dx$ where $P_2(x)$ is the interpolation polynomial of $f(x)$ on the points $x0$, $x1$ and $x2$.

2 Using the Lagrange method:

$$\boldsymbol{L_0(x)} = \frac{\boldsymbol{(x - x_1)(x - x_2)}}{\boldsymbol{(x_0 - x_1)(x_0 - x_2)}}$$

$$\boldsymbol{L_1(x)} = \frac{\boldsymbol{(x - x_0)(x - x_2)}}{\boldsymbol{(x_1 - x_0)(x_1 - x_2)}}$$

$$\boldsymbol{L_2(x)} = \frac{\boldsymbol{(x - x_0)(x - x_1)}}{\boldsymbol{(x_2 - x_0)(x_2 - x_1)}}$$

The points $x0$, $x1$ and $x2$ are equidistant, so $x1 = x0 + h$ and $x2 = x0 + 2h$

3 The generalized Simpson's integral is given by:

$$\boldsymbol{I_{Simp} = \frac{h}{3}[y_0 + y_{2n} + 4\sum_{i=0}^{n-1} y_{2i-1} + 2\sum_{i=1}^{n-1} y_{2i}]}$$

IV MANIPULATIONS

We propose to calculate the definite integral $I = \int_0^2 \sqrt{x}\,dx$, using the 3 methods mentioned above.

1. Write a program that would calculate this integral by the central Lagrange method;

2. Write a program that would calculate this integral using the trapezoid method and dividing the integration interval into **10** elementary intervals;

3. Repeat the execution for **n=30**;

4. Use the matlab **trapz** command to calculate this integral;

5. Repeat this work using the Simpson's method with the same conditions;

6. Use the matlab **quad** command to calculate this integral;

7. Compare the results obtained.