Single-phase circuits and electrical powers

1) *Impedance and admittance*

The impedance of a dipole D, through which a current $i(t)$ flows and which is subjected to a voltage $u(t)$ is define as the ratio:

$$\underline{Z} = \frac{\underline{u}}{\underline{i}} = \frac{\underline{U}}{\underline{I}}$$

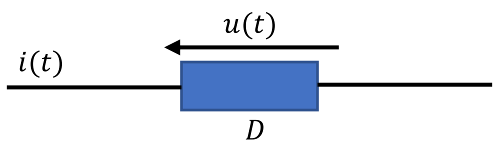

Figure 1: A dipole D

If:

$u(t) = U_m \, Cos(\omega + \varphi)$ and $i(t) = I_m \, Cos(\omega t)$

Then:

$$\underline{Z} = \frac{\underline{u}}{\underline{i}} = \frac{\underline{U}}{\underline{I}} = \frac{U_m e^{j\varphi}}{I_m e^{j0}} = Ze^{j\varphi} = Z\,Cos\varphi + \; jZ\,Sin\varphi = R + j\,X$$

Where :

- $R = Z\,Cos\varphi$: resistance in $[\Omega]$
- $X = Z\,Sin\varphi$: is the reactance in$[\Omega]$
- $|\underline{Z}| = Z = \frac{U_m}{I_m} = \sqrt{R^2 + X^2}$: is the module of $\underline{Z}$ in $[\Omega]$

The admittance of a dipole D is defined as the inverse of its impedance:

$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{R}{R^2 + X^2} - j\,\frac{X}{R^2 + X^2} = G + j\,B$$

Where :

- $G = \frac{R}{R^2+X^2}$: conductance in $[\Omega^{-1}]$
- $B = -\frac{X}{R^2+X^2}$: susceptance in $[\Omega^{-1}]$
- $|\underline{Y}| = Y = \frac{1}{Z} = \frac{1}{\sqrt{R^2+X^2}}$: magnitude of the admittance in $[\Omega^{-1}]$

2) Elements of an electrical circuit

A. Resistor *R*

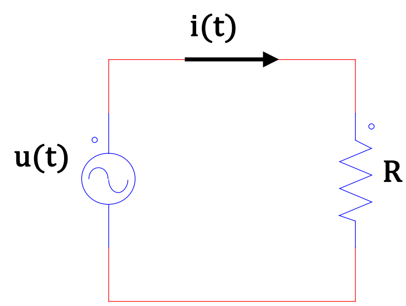

Figure 2: Circuit of a resistor R

Instantaneous relationship:

$$u(t) = Ri(t)$$

In the sinusoidal steady-state this relationship becomes in complex form:

$$\underline{u} = R\underline{i}$$

Hence:

$$\underline{Z}_R = \frac{\underline{u}}{\underline{i}} = R$$

With:

$$\left|\underline{Z}_R\right| = R \text{ and } Arg\left(\underline{Z}_R\right) = 0\, rad$$

If $i(t)$ is chosen as the phase reference (the phase origin):

- **Vector representation**: voltage u and current i are in phase.
- **Graphical representation**: voltage u and current i are in phase.

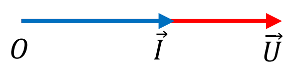

Figure 3: Vector representation

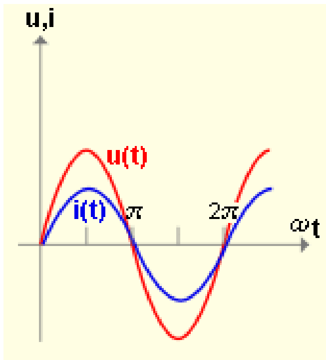

Figure 4: Graphical representation

B. Ideal Inductor L

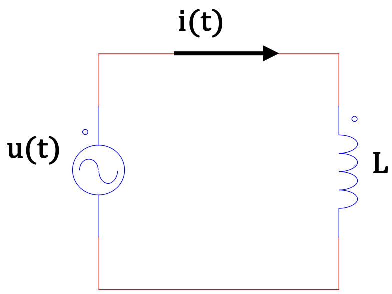

Figure 5: Circuit of an ideal inductor L

Instantaneous relationship:

$$u(t) = L\frac{di(t)}{dt}$$

In the sinusoidal steady-state this relationship becomes in complex form:

$$\underline{u} = jL\omega\underline{i}$$

Hence:

$$\underline{Z_L} = \frac{\underline{u}}{\underline{i}} = jL\omega$$

with:

$$|\underline{Z_L}| = L\omega \text{ And } Arg(\underline{Z_L}) = \frac{\pi}{2}\, rad$$

- **Vector representation**: the current i **lags** the voltage u by $\frac{\pi}{2}$radians.
- **Graphical representation**: the voltage waveform u **leads** the current waveform i by a quarter period.

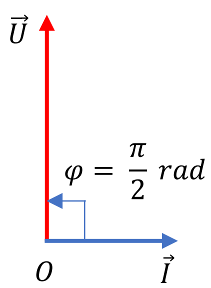

Figure 6: Vector representation

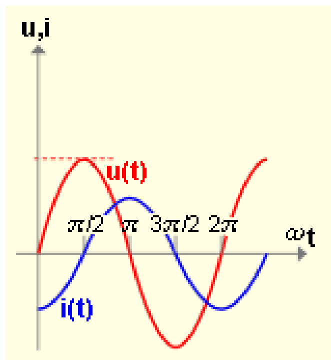

Figure 7: Graphical representation

C. Capacitor C

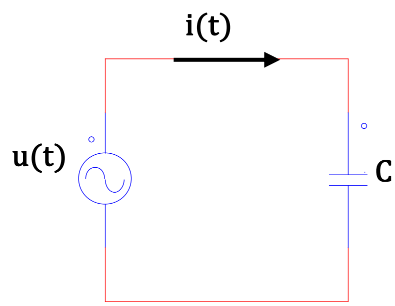

Figure 8: Circuit pf a capacitor C

Instantaneous relationship:

$$u(t) = \frac{1}{C}\int i(t)dt$$

In the sinusoidal steady-state, the complex form is:

$$\underline{u} = \frac{1}{jC\omega}\underline{i}$$

Hence:

$$\underline{Z}_C = \frac{\underline{u}}{\underline{i}} = \frac{1}{jC\omega} = -j\frac{1}{C\omega}$$

With:

$$\left|\underline{Z}_C\right| = \frac{1}{C\omega} \text{ And } Arg\left(\underline{Z}_C\right) = -\frac{\pi}{2}\ rad$$

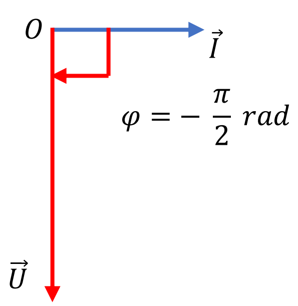

Figure 9:Vector representation

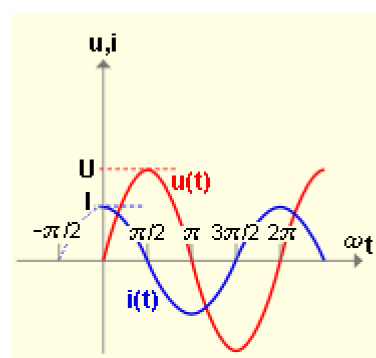

Figure 10: Graphical representation

- **Vector representation**: the current i **lead s** the voltage u by $\frac{\pi}{2}$ radians.
- **Graphical representation**: the current waveform i reaches its maximum a quarter period before the voltage waveform u.

3) Fundamental laws of electricity

A. Generalized Ohm's law

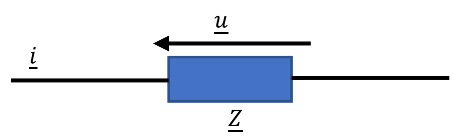

Figure11: A dipole D with $\underline{Z}$ impedance

The generalized Ohm's law in complex form relates the voltage $\underline{u}$ or $\underline{U}$ to the current $\underline{i}$ or $\underline{I}$ through the impedance $\underline{Z}$ which can be complex.

$$\underline{u} = \underline{z}\,\underline{i} \; ou \; \underline{U} = \underline{Z}\,\underline{I}$$

This relationship holds in sinusoidal steady-state conditions where voltage and current are represented as complex numbers.

B. Kirchhoff's laws

- **Kirchhoff's voltage law (KVL)**

The phasor sum of the voltages around any closed loop is zero. This law arises from the conservation of energy in an electrical circuit.

$$\sum_{k=1}^{n} \underline{u}_k = 0$$

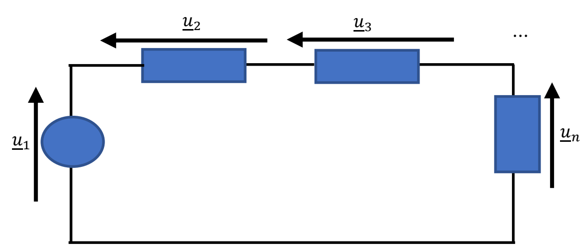

Figure 12: Electrical circuit with many dipoles

- **Kirchhoff's current law (KCL)**

The phasor sum of the voltages around any closed loop is zero. This law arises from the conservation of electric charge.

$$\sum_{k=1}^{n} \underline{i}_k = 0$$

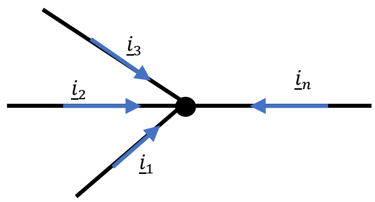

Figure 13: A node with many currents

- **Voltage and current divider**

$$\underline{u}_k = \frac{\underline{Z}_k}{\underline{Z}_{eq}}\underline{u} \qquad \text{Where:} \qquad \underline{Z}_{eq} = \sum_{k=1}^{n}\underline{Z}_k$$

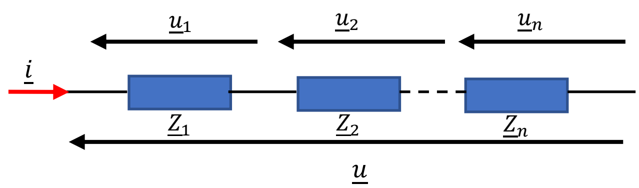

Figure 14: Voltage divider

And: $\underline{i}_k = \frac{\underline{Y}_k}{\sum_{k=1}^{n}\underline{Y}_k}\underline{i}$ Where: $\sum_{k=1}^{n}\underline{Y}_k = \sum_{k=1}^{n}\frac{1}{\underline{Z}_k}$

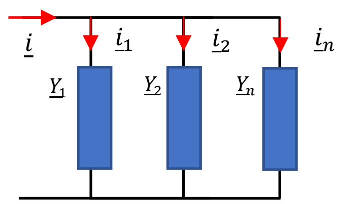

Figure15: Currents divider

4) Association of dipoles (impedances)

A. In series

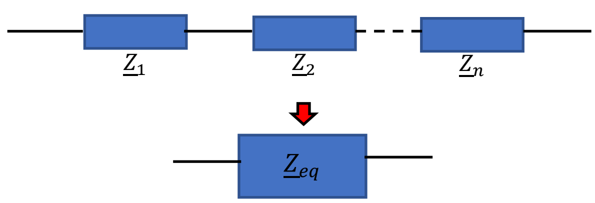

Figure16: Impedance equivalent in series

$$\underline{Z}_{eq} = \underline{Z}_1 + \underline{Z}_2 + \cdots + \underline{Z}_n = \sum_{k=1}^{n}\underline{Z}_k$$

B. In parallel

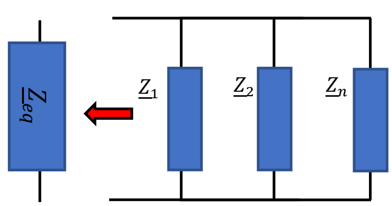

Figure17: Impedance equivalent in parallel

$$\frac{1}{\underline{Z}_{eq}} = \frac{1}{\underline{Z}_1} + \frac{1}{\underline{Z}_2} + \cdots + \frac{1}{\underline{Z}_n} = \sum_{k=1}^{n}\frac{1}{\underline{Z}_k} = \sum_{k=1}^{n}\underline{Y}_k$$

C. Special cases

Elements	$\underline{Z}$	$\lvert\underline{Z}\rvert$	$\varphi(rad)$
R	R	R	0
L	$jL\omega$	$L\omega$	$\pi/2$
C	$-j/C\omega$	$1/C\omega$	$-\pi/2$
RL in series	$R + jL\omega$	$\sqrt{R^2 + L\omega^2}$	$tan^{-1}(\frac{L\omega}{R})$
RC in series	$R - J\frac{1}{C\omega}$	$\sqrt{R^2 + (-\frac{1}{C\omega})^2}$	$tan^{-1}(-\frac{1}{RC\omega})$
LC in series	$j(L\omega - \frac{1}{C\omega})$	$\left\lvert L\omega - \frac{1}{C\omega}\right\rvert$	$\pm\pi/2$

5) Impedances triangle

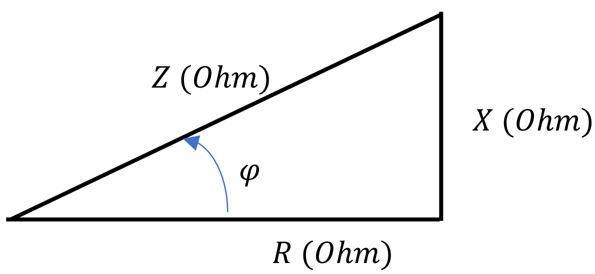

Figure 18: Impedances triangle

Electrical power in sinusoidal regime

1. Instantaneous power

We Consider a dipole D in receiver convention through which a current $i(t)$ flows and across which voltage $u(t)$ is applied.

The instantaneous power dissipated in the dipole $\boldsymbol{D}$ is expressed as:

$$P(t) = u(t)\ i(t)$$

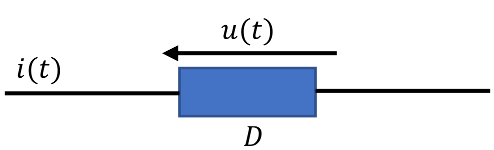

Figure 19: A dipole D

2. Instantaneous power in sinusoidal mode

For a sinusoidal steady-state the instantaneous power can be written as:

$$P(t) = U_{rms}I_{rms}Cos\varphi + U_{rms}I_{rms}\, Cos(2\omega t + \varphi)$$

- The first term represents the average (active) power.
- The second term the fluctuating power exchanged periodically between the source and the load.
- The fluctuating power (variable component of $P(t)$) is therefore:

$$P_f(t) = U_{rms}I_{rms}\, Cos(2\omega t + \varphi)$$

3. Active power

- The active power $\boldsymbol{P}$ is the average value of $P(t)$ over one period:

$$P = \frac{1}{T}\int_0^T P(t)\, dt = U_{rms}I_{rms}Cos\varphi \quad \text{[W]}$$

Note:

- For a pure resistor $\boldsymbol{R}$

$$U_{rms} = RI_{rms} \Rightarrow P = R\, I_{rms}^2 \qquad \text{[W]}$$

- For an ideal inductor $\boldsymbol{L}$ or capacitor $\boldsymbol{C}$

$$\varphi = \pm\frac{\pi}{2}\, rad \Rightarrow P = 0\, rad \quad \text{[W]}$$

4. Reactive power

The reactive power $\boldsymbol{Q}$ is the amplitude of the alternating component of $P(t)$:

$$Q = U_{rms}I_{rms}\, Sin\varphi \qquad \text{[Var]}$$

Note:

- ❖ For a resistor R:

$$\varphi = 0 \Rightarrow Q = 0 \quad \text{[Var]}$$

- ❖ For an ideal inductor L:

$$U_{rms} = X_L\, I_{rms} \Rightarrow Q = X_L I_{rms}^2 \text{ [Var]}$$

Where: $X = L\omega$

- ❖ For a capacitor C: $Q = -X_C I_{rms}^2$ [Var]

Where :

$$X_C = \frac{1}{(C\omega)}$$

5. Apparent Power

The apparent power S represents the total power supplied by the source:

$$S = \sqrt{P^2 + Q^2} = U_{rms} I_{rms} = Z I_{rms}^2 \qquad \text{[VA]}$$

It can be expressed in complex form:

$$\underline{S} = P + j\,Q$$

6. Power factor

The power factor P_F measures the efficiency of power transfer to the load

$$P_F = Cos\,\varphi = \frac{P}{S} \qquad Where: \; 0 < P_F < 1$$

A low power factor indicates a large reactive component (inductive or capacitive load), leading to inefficient power transfer.

7. Power triangles

The power components can be represented geometrically in the power triangle

$$S^2 = P^2 + Q^2$$

P: Active power (horizontal axis)

Q: Reactive power (vertical axis)

S: Apparent power (hypotenuse)

φ: Phase shift between voltage and current

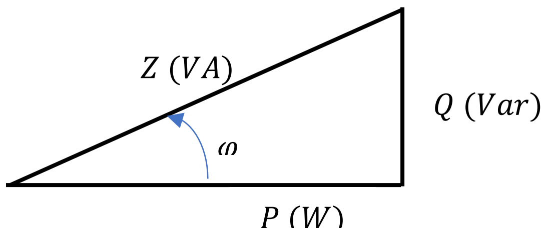

Figure 20: Powers triangle

8. Boucherot's Theorem

For a single-phase installation with several loads connected in parallel or series:

- $P_T = \sum_{k=1}^{N} P_k$
- $Q_T = \sum_{k=1}^{N} Q_k$
- $S_T \neq \sum_{k=1}^{N} S_k$

However, in complex form:

- $\underline{S}_T = \sum_{k=1}^{N} \underline{S}_k$

And the total apparent power is:

- $S_T = \sqrt{P_T^2 + Q_T^2}$

The global power factor is then:

- $P_F = Cos\ \varphi_T = \frac{P_T}{S_T}$

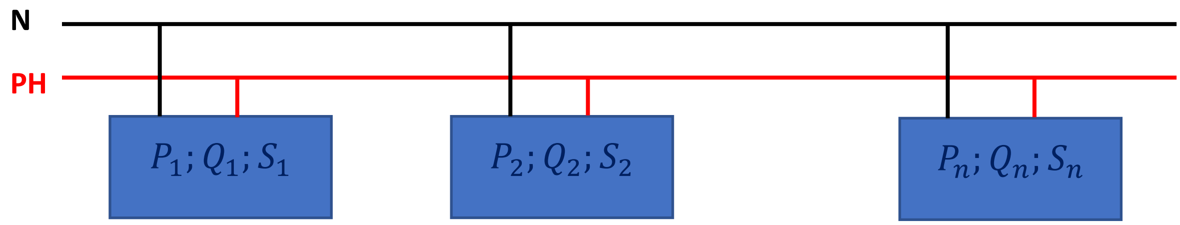

Figure 21 : Single phase installation