Chapter 06

Sampling and Discrete Signals

6.1 Introduction

A discrete signal is a signal obtained by expressing information at point values of time $x(t_i)$; it therefore constitutes a digital sequence. In practice, discrete signals are most often obtained by the operation called sampling.

6.2 Sampling

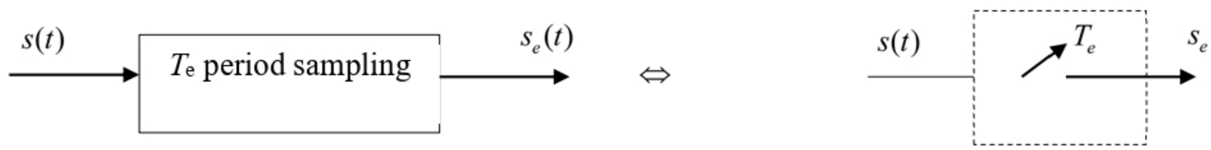
Sampling an analog signal means replacing this continuous signal with sequences (samples) of these values for well-defined instants, thus transforming it into a set of discrete values. $s(nT_e)$ This process is carried out using a sampler, which takes samples at times generally spaced at a constant time. T_e.

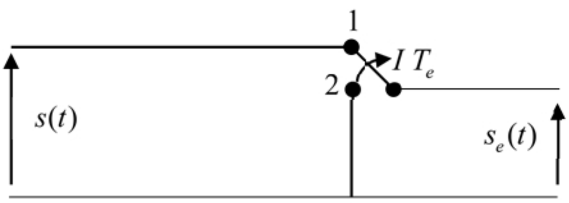
n : positive integer variable, $n \in \mathbb{N}$;

T_e : is the sampling period (By definition $T_e > 0$) ; nT_e : are the sampling times.

6.2.1 Principle of sampling

The idea is to use an ideal switch that is closed for a very short time, then opened for a T_e (sampling period). The following two figures show the modeling of the sampling operation by an ideal switch.

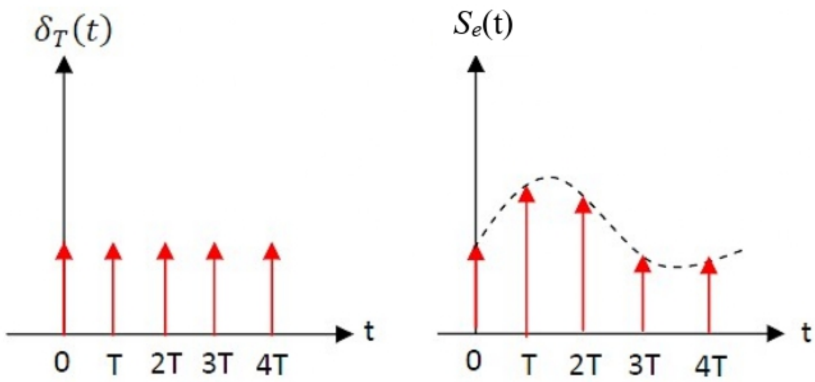




$I = 1 \Rightarrow s_e(t) = s(t)$

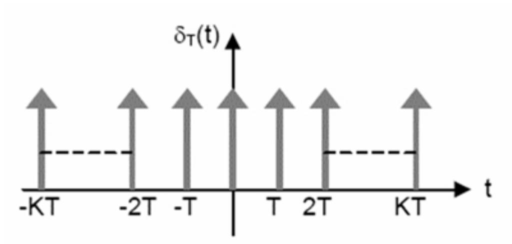
$I = 2 \Rightarrow s_e(t) = 0$

Applying the sampling function to the signal $y(t)$ then produces the sampled signal $y^*(t)$ defined as follows:



The operation described above can be formalized as follows. We first define $\delta(t)$, the sampling function at times nT_e given by:

$$\delta_{T_e}(t) = \sum_{n=0}^{+\infty} \delta(t - nT_e)$$



$\delta_{T_e}(t)$: is a periodic signal of period $T_e \Rightarrow$ it admits a complex Fourier series development of the form:

$$\delta_{T_e}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{+j2\pi n f_e t}$$

With:

$$c_n = \frac{1}{T_e} \int_{-\frac{Te}{2}}^{+\frac{T_e}{2}} \delta_{T_e}(t) e^{-j2\pi n f_e t} dt$$

$$c_n = \frac{1}{T_e} \int_{-\frac{Te}{2}}^{+\frac{T_e}{2}} \delta(t) e^{-j2\pi n f_e t} dt$$

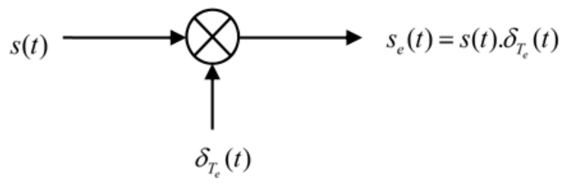
On the property of the Dirac momentum:

$$\int_{-\infty}^{+\infty} \delta(t) X(t) dt = X(0)$$

So:

$$c_n = \frac{1}{T_e} \Rightarrow \delta_{T_e}(t) = \sum_{-\infty}^{+\infty} \frac{1}{T_e} e^{+j2\pi n f_e t}$$

A sampled signal $s_e(t)$ is represented by:



$$s_e(t) = s(t).\sum_{-\infty}^{+\infty} \delta_{T_e}(t - nT_e)$$

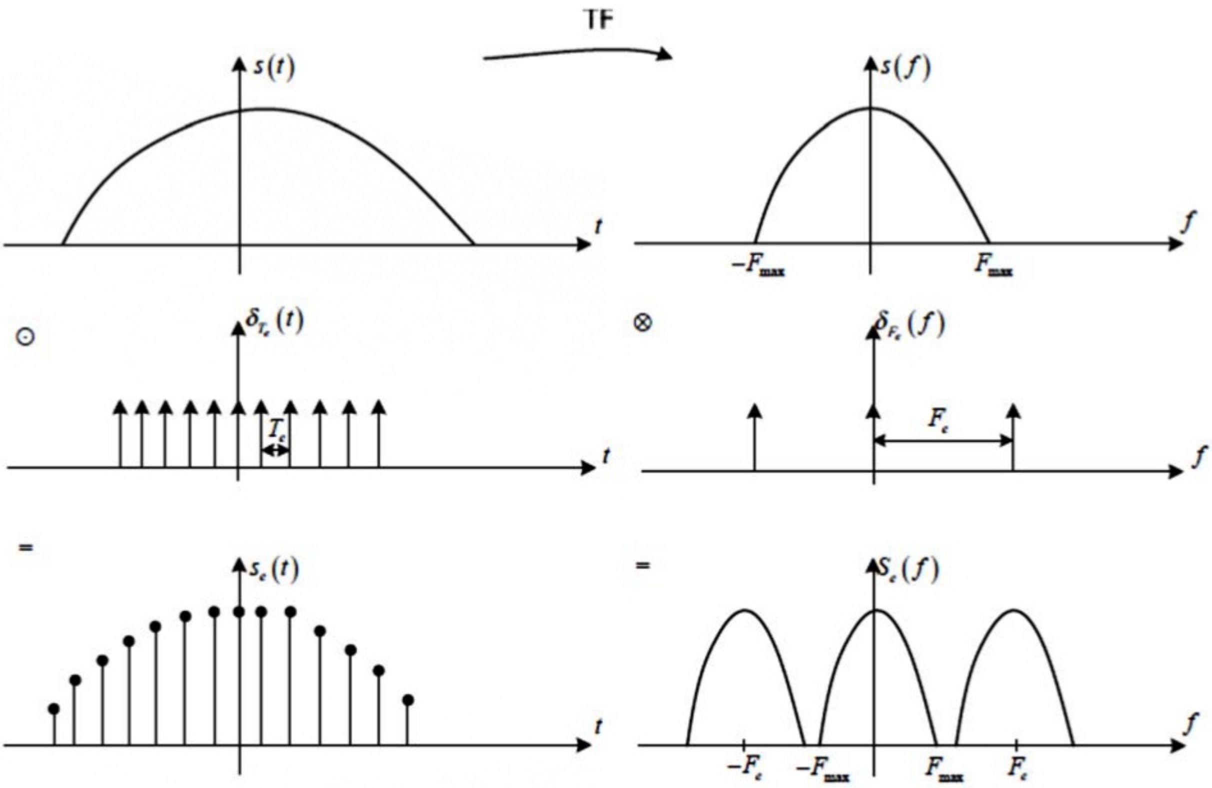
$$S_e(f) = s(f).TF\left\{\sum_{-\infty}^{+\infty} \delta_{T_e}(t - nF_e)\right\}$$

$$= \frac{1}{T_e} S(f) * \sum_{-\infty}^{+\infty} \delta(f - nF_e)$$

$$= \frac{1}{T_e} \sum_{-\infty}^{+\infty} X(f - nF_e)$$

$$= F_e.\sum_{-\infty}^{+\infty} X(f - nF_e)$$

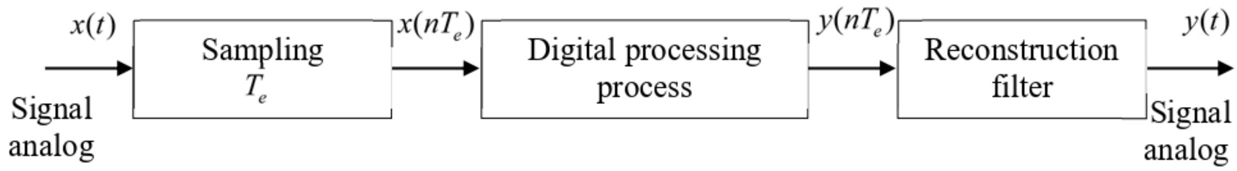
With : $\frac{1}{T_e} = f_e$



- Time Sampling $\xrightarrow{TF}$ periodicity of the spectrum.
- The spectrum of $X_e(f)$ is that of $X(f)$ "periodized" with a frequency period F_e .
- Sampling in the time domain results in a period "periodization" F_e in the frequency domain.

The spectrum of the sampled signal is formed by a sequence of the spectrum of the analog signal (the spectrum of the analog signal repeats periodically on the frequency axis with a period $\frac{1}{T_e} = f_e$.

6.2.2 Shannon's Theorem



By studying the block diagram of a processing and control chain, we see that two questions can arise.

1) What should be the value of T_e to have a sampled signal most faithfully representing the signal $x(t)$.
2) How should the reconstruction filter be chosen to obtain an analog signal? $y(t)$ as representative as possible.

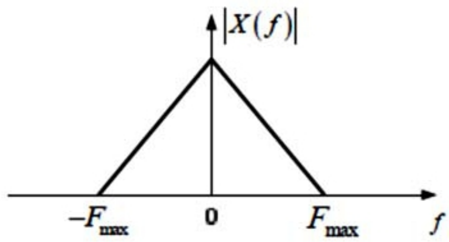
As for the first question, the solution was given by the Shannon theorem, it is necessary to respect the condition:

$$F_e \geq 2F_{\max}; \quad T_e \leq \frac{1}{2}T_{\max}$$

$F_{\max}$: being the maximum frequency of analog signal.

The signal must have a bounded spectrum so that it can be reconstructed from its sampled version provided that the maximum frequency $F_e \geq 2F_{\max}$.

It is considered that $x(t)$ is a real signal whose spectrum is bounded in frequency, with maximum frequency $F_{\max}$.



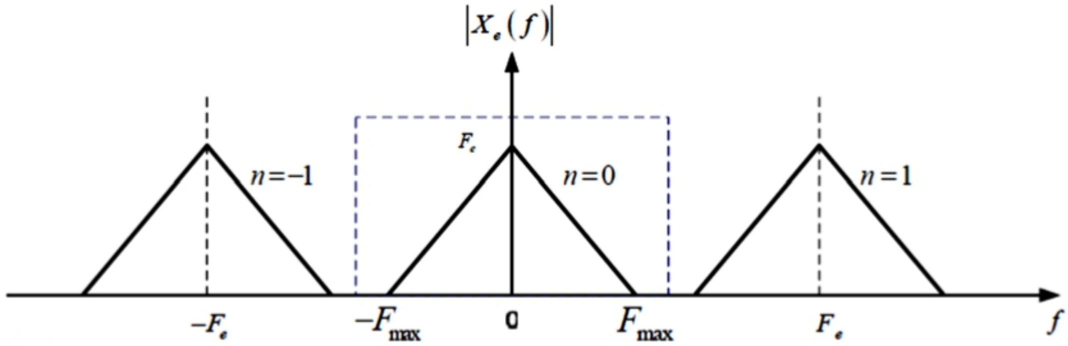
$$\forall |f| > F_{\max}; \quad |X(f)| = 0$$

Question: What happens to the spectrum? $X_e(f)$ depending on F_e ?

$$X_e(f) = F_e . \sum_{n=-\infty}^{n=+\infty} X(f - nF_e)$$

1st case

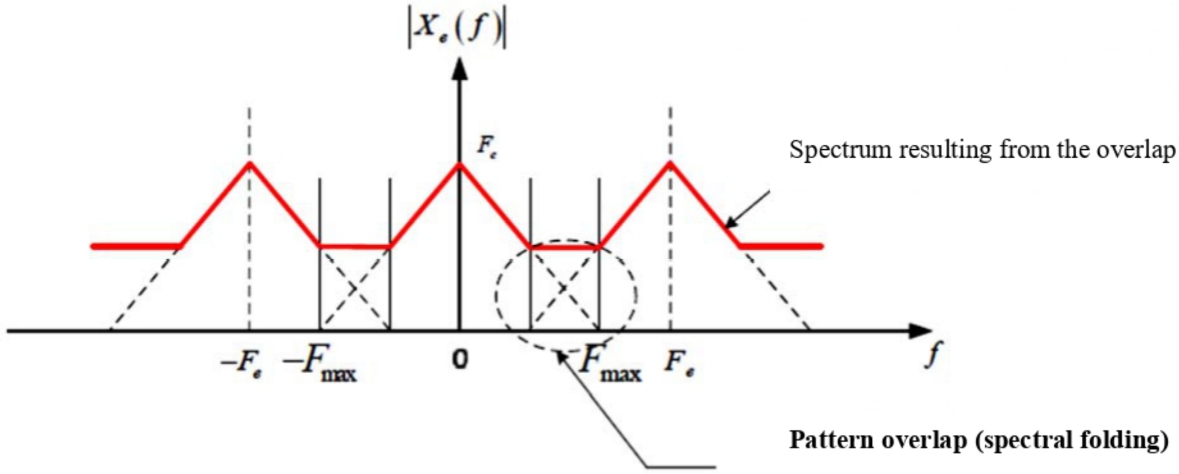
$$F_e \geq 2F_{\max}$$



The main motive $(n=0)$ is equal to the spectrum of $x(t)$. Since the patterns are disjoint, we can extract $X(f)$ thanks to an ideal low-pass filter and therefore fully reconstruct the signal $x(t)$ from the knowledge of its sampled $x_e(t)$.

<u>2nd case</u>

$F_e < 2F_{max}$



<u>Question</u> : what is the condition on F_e so that from the sampled signal $x_e(t)$ we can completely rebuild $x(t)$?

$F_e \geq 2F_{max}$ no spectrum overlap $\rightarrow$ extraction of $X(f)$ by ideal low-pass filtering.

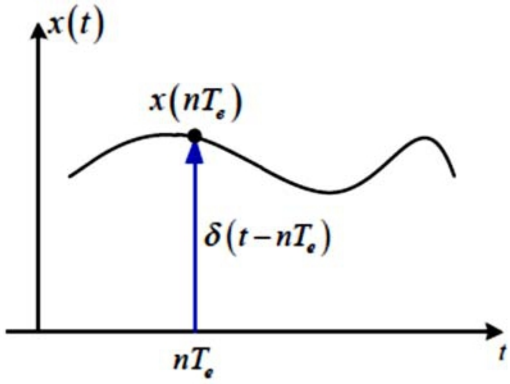
$F_e < 2F_{max}$ spectrum folding $\rightarrow$ unable to recover $X(f)$ by filtering.

Therefore, for the periodic repetition of the spectrum of $x_e(t)$ does not distort the spectrum $X(f)$ repeated, it is necessary and sufficient that $F_e \geq 2F_{max}$ (apply Shannon's Theorem).

6.2.3 Ideal sampling

Ideal sampling involves using an infinitely short pulse to extract the instantaneous value $x(nT_e)$ right now nT_e.

So this is the application of the distribution $\delta(t - nT_e)$ to the continuous signal $x(t)$:



By definition:

$$x_e(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - nT_e)d\tau$$

We note that:

$$x(t) \otimes \delta(-t) = \int_{-\infty}^{+\infty} x(\tau)\delta(\tau - t)d\tau$$

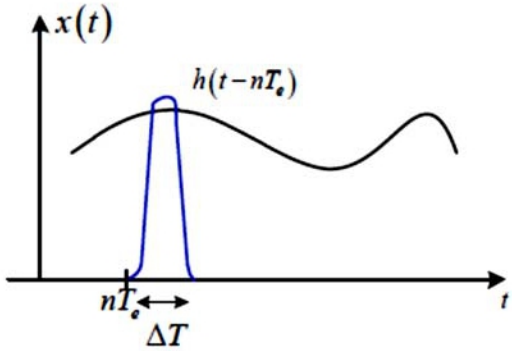
Then deduce:

$$x(nT_e) = x(t) \otimes \delta(-t)\Big|_{t=nT_e}$$

The sampler is similar to an impulse response filter. $\delta(-t)$.

6.2.4 Real sampling

In practice, we do not have an infinitely short pulse and the sampler is similar to an impulse response filter. $h(-t)$.



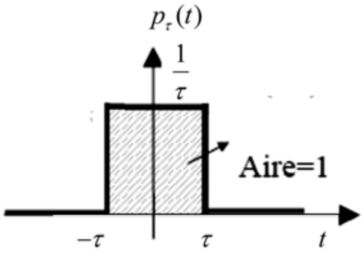
$$x(nT_e) = x(t) \otimes h(-t)\Big|_{t=nT_e}$$

$$x(nT_e) = \left[x(t) \otimes h(-t)\right]\delta(t - nT_e)$$

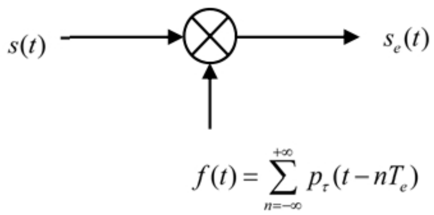
Hence the expression of the real sampled signal:

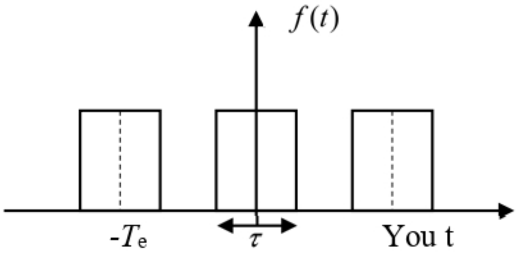
$$x_e(t) = \left[x(t) \otimes h(-t)\right]\sum_{-\infty}^{+\infty}\delta(t - nT_e)$$

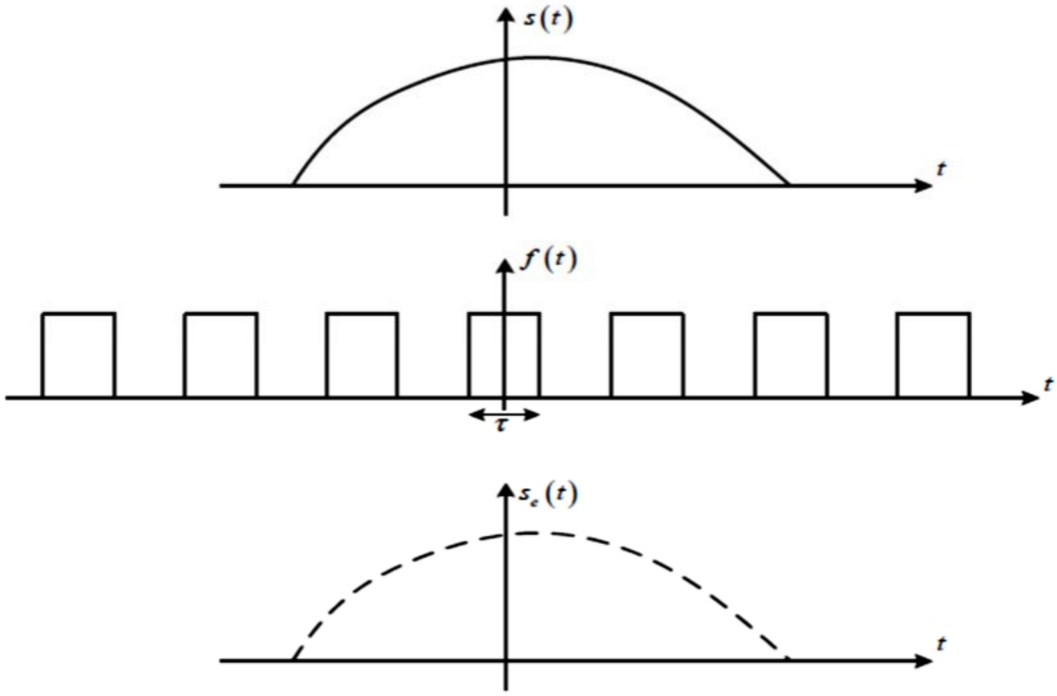
Knowing that the Dirac comb is not a physically realizable function. So sampling with this type of function cannot be realizable. On the other hand, we know that a Dirac can be assimilated to:



With : $\lim_{\tau\to 0}\frac{1}{\tau}p_\tau(t)=\delta(t)$, A real periodic sampling of an analog signal $s(t)$ is obtained, by multiplying is signaled by a sampling function $f(t)$ periodic rectangular pulse.







$$s_e(t)=s(t).p_\tau(t-nT_e)$$
$$=s(t).\left[p_\tau(t)\otimes\delta(t-nT_e)\right]$$

$$\xrightarrow{F} S_e(f)=S(f)*\left[\tau\sin c(\pi f\tau).F_e\sum_{n=-\infty}^{+\infty}\delta(f-nF_e)\right]$$

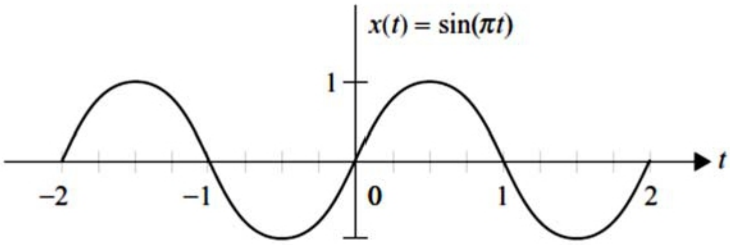
$$S_e(f)=F_e\tau S(f)*\sum_{n=-\infty}^{+\infty}\sin c(nF_e\tau)\delta(f-nF_e)$$

$$S_e(f)=F_e\tau\sum_{n=-\infty}^{+\infty}\sin c(nF_e\tau)S(f-nF_e)$$

The spectrum of a sampled signal actually consists of a series of patterns $S(f)$ which repeats periodically on the frequency axis with a period F_e. These spectral patterns are weighted by a factor that depends on the pulse density. $F_e\tau$ and which varies as a function of n according to a law in $\sin c$.

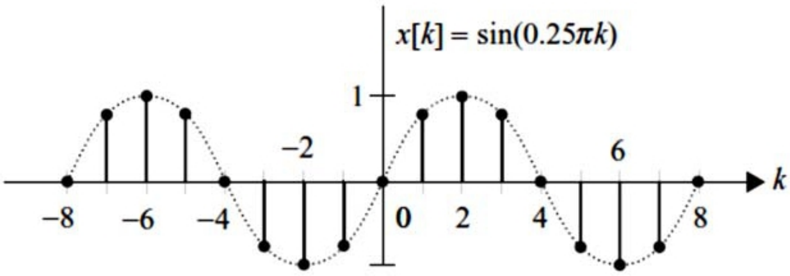
Example 6.1:

We consider the signal $x(t) = \sin(\pi t)$ from the following figure:



1- Discretize this signal for a sampling period: $T = 0.25\text{s}$?, with : $-8 \le k \le +8$

2- Graphically represent the sampled signal.



Example 6.2:

Let the signal be:

$s(t) = 4f_0 \sin c(4\pi f_0 t) + f_0 \sin c^2(\pi f_0 t)$

This signal is ideally sampled at a frequency F_e.

1- Calculate the Fourier transform $S(f)$ of this signal.

3- Graphically represent the signal $S(f)$.

4- Determine the minimum sampling frequency allowing the exact reconstruction of the signal.

5- Plot the spectrum of the sampled signal for a sampling frequency $F_e = 6f_0$.

6.2.5 Quantification

The quantization operation consists of assigning a binary number to any value taken from the signal during sampling.

This operation is performed by the Analog to Digital Converter (ADC).

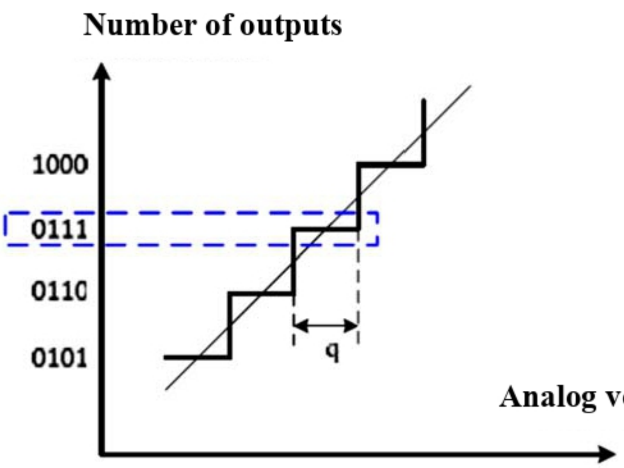
Each voltage level is coded on *p* bits, each bit being able to take two values (0 or 1).

So a converter to p bits owns 2^p quantization levels.

Consider a 4-bit CAN, so there are only $2^4 = 16$ possible values attributable to all values taken during sampling.

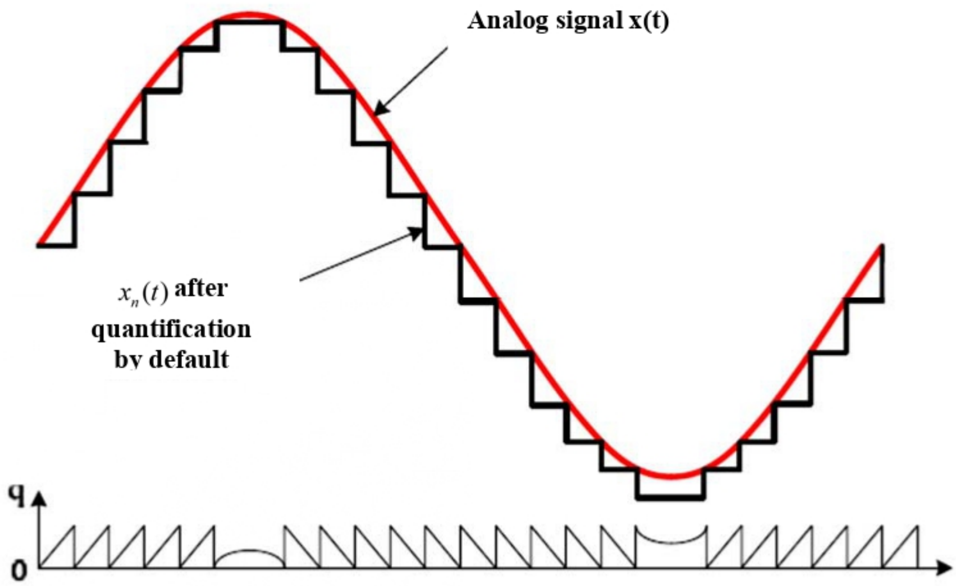
The operation is therefore carried out with a loss of information that is all the greater as p is small.

The diagram below represents part of the transfer characteristic of a 4-bit converter; at all voltage levels of the same step, the converter therefore corresponds to a single binary number:



q is the quantization step: it corresponds to the smallest voltage variation that the converter can code. We can see that the more q is low, the better the coding accuracy will be.

For default quantization, where $x_n = nq$ if x is understood between nq And $(n+1)q$, the error made called quantization noise is given in the graph below:



Quantization noise:

$\varepsilon(t) = x(t) - x_n(t)$

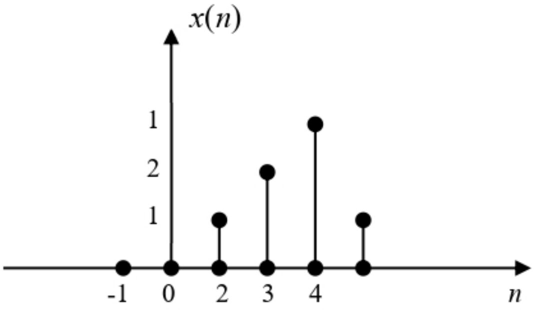
6.3 Discrete-time signals

Number sequence:

$$x[n] = \{x(n)\} \quad ,-\infty < n < \infty$$

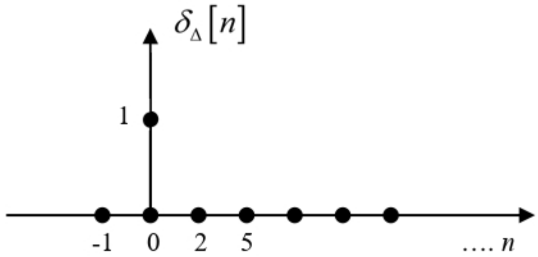
$x(n)$ = nth sample of the sequence.

Example 6.3:



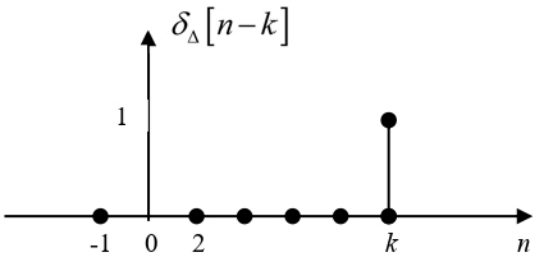
6.3.1 The Kronecker impulse

$$\delta_{\Delta}[n] = \begin{cases} 1 \; if \; n = 0 \\ 0 \; if \; n \neq 0 \end{cases}$$



The shifted Kronecker impulse:

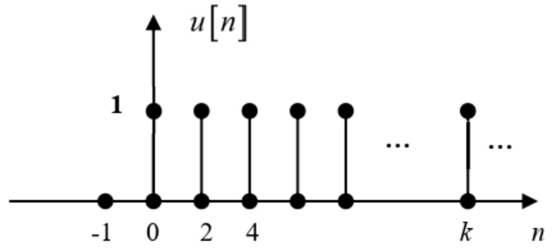
$$\delta_{\Delta}[n-k] = \begin{cases} 1 \; if \; n = k \\ 0 \; if \; n \neq k \end{cases}$$



6.3.2 The unit level

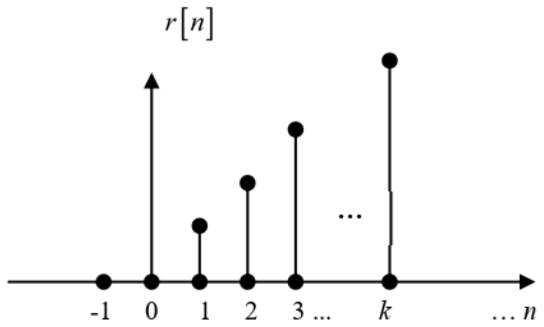
$$u[n] = \begin{cases} 1 \; if \; n \geq 0 \\ 0 \; if \; n < 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$



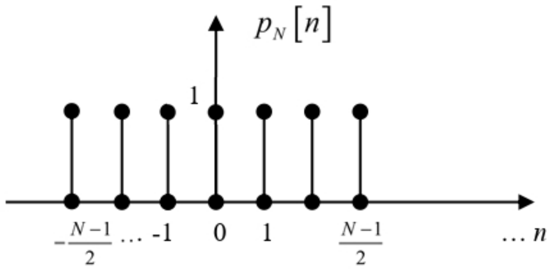
6.3.3 Ramp Function

$$r[n] = \begin{cases} n \text{ if } n \geq 0 \\ 0 \text{ if } n < 0 \end{cases}$$



6.3.4 Exponential function

$$p_N[n] = \begin{cases} 1 \text{ if } |n| \leq \frac{N-1}{2} \\ 0 \text{ if } |n| > \frac{N-1}{2} \end{cases}$$

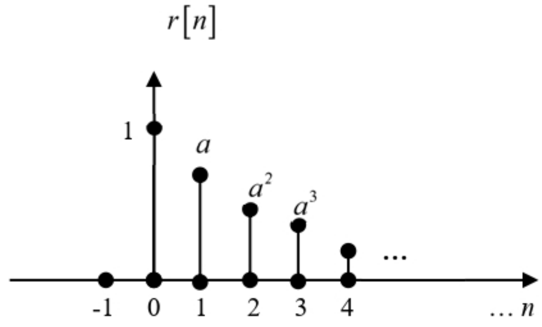


$$p_N[n] = u(n + \frac{N-1}{2}) - u(n - \frac{N+1}{2})$$

6.3.5 Exponential function

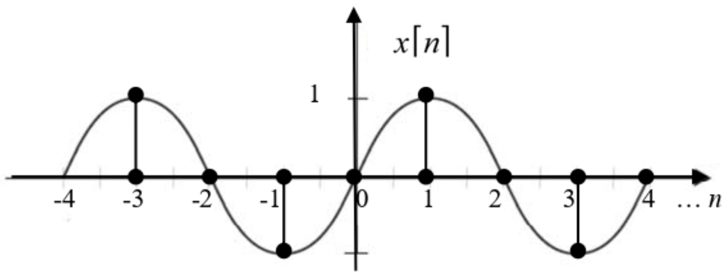
$$x[n] = \begin{cases} a^n \text{ if } n \geq 0 \\ 0 \text{ if } n < 0 \end{cases}$$

With : $a = e^{-\alpha T}$



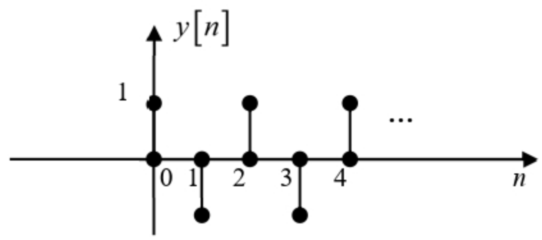
6.3.6 Sinusoidal sequence

$$x[n] \ = \begin{cases} \sin(\frac{n\pi}{2}) \ if \ n \geq 0 \\ 0 \ if \ n < 0 \end{cases}$$



6.3.7 Alternate Suite

$$y[n] \ = \begin{cases} (-1)^n \ if \ n \geq 0 \\ 0 \ if \ n < 0 \end{cases}$$



Example 6.4:

Consider the following discrete signal:

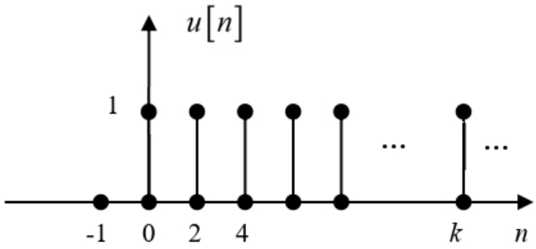
$$y(n) \ = \{y(-2) = 1 \, , y(-1) = -1.5 \, , y(0) = 3 \, , y(1) = 0 \, , y(2) = -0.7 \, , y(3) = 2.3\}$$

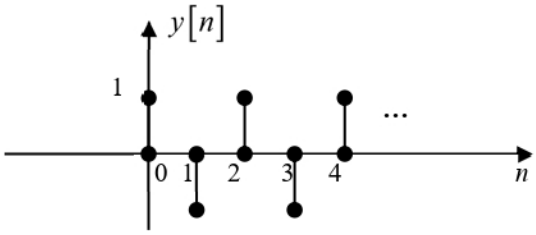
1- Graphically represent the signal $y(n)$.

2- Find the mathematical expression of the signal $y(n)$ according to the Kronecker impulse $\delta_{\Delta}(n)$.

Example 6.5:

Let the following two discrete signals be:





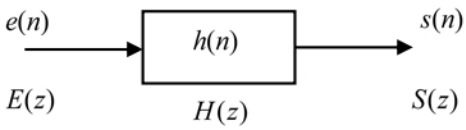
1- Find the mathematical expression of the signal $g[n] = u[n] + y[n]$.

2- Graphically represent the signal $g[n]$.

6.4 Convolution product

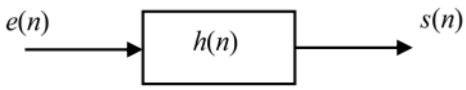
It is given by:

$$s(n) = e(n) * h(n)$$
$$= \sum_{i=-\infty}^{+\infty} e(i).h(n-i)$$



Example 6.6:

Let the digital system be $h(n)$ following :



With $: h(n) = e^{-b.n.T_0}.u(n)$ with $b > 0\,; T_0$: sampling period, The input to this system is given by: $e(n) = u(n) - u(n-3)$.

1- Found the exit $s(n)$ of the system $h(n)$.

6.5 Z Transform

Transformed it into Z is used to simplify the operations of difference equations. It is used as a tool in the analysis and design of discrete sampled systems, it allows to simplify the solution of a discrete system problem by converting the difference equations of a system into algebraic equations and convolving them into multiplication.

It plays the same role as the Laplace transform for continuous systems.

Continuous systems→ Laplace transform (L).

Discrete systems→ Transformed into Z (Z).

6.5.1 Definition

Transformed it into Z of a sequence $x[n] = x_0, x_1, x_2, ..., x_n$ is defined as the series $X(z)$ calculated as follows:

$$X(z) = \sum_{n=0}^{+\infty} x[nT_e].z^{-n}$$

Or:

$$X(z) = \sum_{n=0}^{+\infty} x[n].z^{-n} = x_0 + x_1 z^{-1} + x_2 z^{-2} + \ldots + x_n z^{-n}$$

Where:

- The variable z^{-1} is a delay operator,
- T : sampling time.
- The relationship with the Laplace transform $z = e^{p.T_e}$

z : is a complex variable.

with : $z = r.e^{j.w}$

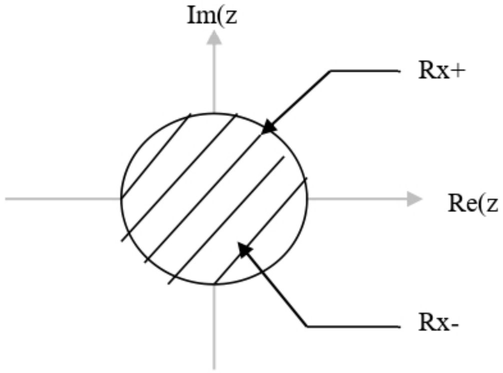
SO : $X(r.e^{j.\omega}) = \sum_{n=0}^{+\infty} x[n].r^{-n}.e^{-j.n.\omega}$

Transformed it into Z can be interpreted as the Fourier transform of $x[n]$ weighted by an exponential sequence (r^{-n}).

Transformed it into Z converges if: $\sum_{n=-\infty}^{+\infty} \left| x[n].r^{-n} \right| < \infty$

- The set of values of z for which the transformation into Z converges and called the domain of convergence.
- If the Fourier transform of the discrete system ($r = 1$) does not converge, the transform into Z can converge (thanks to the term r^{-n}).
- The domain of convergence is generally a ring centered on the origin: $R_{x^-} < |z| < R_{x^+}$

  Or R_{x^-} can be reduced to 0 and R_{x^+} can be equal to the ∞ .



So the system is stable if the poles of the transfer function are inside the circle.

6.5.2 Z-transform of standard discrete signals

The following identities will be used to define the transform into Z standard signals:

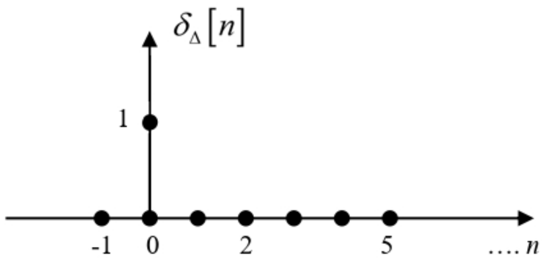
$\sum_{n=0}^{k} a^n = \frac{1-a^{k+1}}{1-a}$; with $a \neq 1$.

$\sum_{n=0}^{k} a^n = \frac{1}{1-a}$; with $|a| < 1$

1) Kronecker's impulse

$$\delta_\Delta[n] = \begin{cases} 1 \ if \ n = 0 \\ 0 \ if \ n \neq 0 \end{cases} \Rightarrow \delta_\Delta(z) = \sum_{n=0}^{+\infty} \delta_\Delta(n).z^{-n}$$

$$= 1 + 0.z^{-1} + 0.z^{-2} + ... = 1$$



2) The unit level

$$u[n] = \begin{cases} 1 \ if \ n \geq 0 \\ 0 \ if \ n < 0 \end{cases}$$

$$= \sum_{k=0}^{+\infty} \delta[n-k]$$

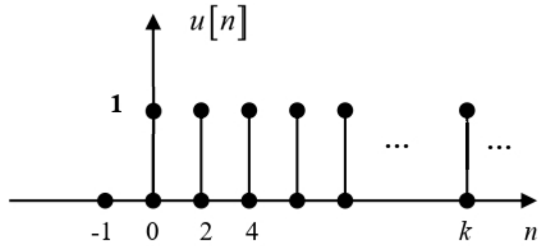
$$X(z) = \sum_{n=0}^{+\infty} u(n).z^{-n}$$

$$= 1 + 1.z^{-1} + 1.z^{-2} + ...$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

Using the identities already seen:

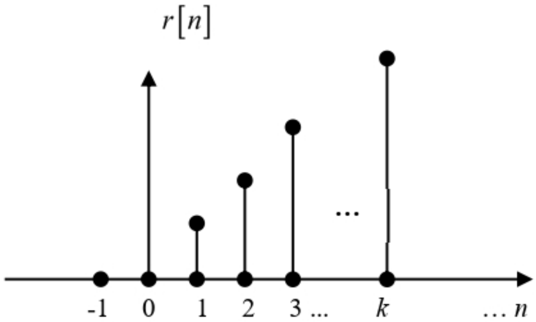
$$U(z) = \frac{1}{1 - z^{-1}}$$
$$= \frac{z}{z - 1}; \quad if \quad |z| < 1$$



3) Ramp Function

$$r[n] = \begin{cases} n \ if \ n \geq 0 \\ 0 \ if \ n < 0 \end{cases}$$

$$R(z) = \sum_{n=0}^{+\infty} n.z^{-n} = 1 + 2z^{-2} + 3z^{-3} + \ldots$$
$$= \frac{z^{-1}}{\left(1 - z^{-1}\right)^2}$$
$$= \frac{z}{\left(z - 1\right)^2}$$



4) Exponential function

$$x(t) = e^{-\alpha t} u(t) \qquad \rightarrow \text{(Continuous case)}$$

$$x(n) = e^{-\alpha nT} = \left(e^{-\alpha T}\right)^n$$

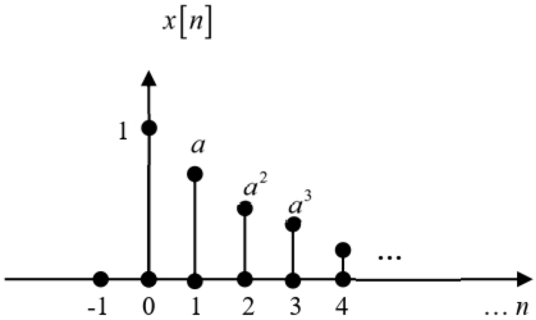
With : $a = e^{-\alpha T}$

$$x[n] = \begin{cases} a^n \ if \ n \geq 0 \\ 0 \ if \ n < 0 \end{cases}$$

$$X(z)=\sum_{n=0}^{+\infty}a^{n}.z^{-n}=\sum_{n=0}^{+\infty}\left(a.z^{-1}\right)^{n}$$
$$=1+az^{-2}+az^{-3}+\ldots+a^{n}z^{-n}+\ldots$$

Using the already established identities:

$$X(z)=\frac{1}{1-az^{-1}}$$
$$=\frac{z}{z-a}$$



6.5.3 Properties of the Z Transform

1) Linearity

Let the two discrete functions be $g(n), h(n)$:

So:

$$Z\{\alpha.g(n)+\beta.h(n)\}=Z\{\alpha.g(n)\}+Z\{\beta.h(n)\}$$
$$=\alpha.G(z)+\beta H(z)$$

Example 6.7:

Let the causal sequence be:

$x(n)=2u(n)+4\delta(n);\quad n=0,1,2\ldots$

1- Calculate the Z transform of this sequence.

2) Multiplication with an exponential a^n

$$Z\left\{a^{n}x(n)\right\}=X(az)$$

$$\sum_{n=0}^{\infty}a^{-n}x(n)z^{n}=\sum_{n=0}^{\infty}x(n)\left(az\right)^{-n}$$
$$=X(az)$$

Example 6.8:

Let the exponential sequence be:

$g(n)=e^{-\alpha nT},\quad n=0,1,2,\ldots$

1- Calculate the Z transform of this signal.

3) Time delay

$Z\{x(n-k)\} = z^{-k}X(z)$

Example 6.9:

We consider the following discrete signal:

$$x(n) = \begin{cases} 4, & n = 2,3... \\ 0, & n = 0,1 \end{cases}$$

1- Calculate the Z transform using the delay theorem.

***4) Time advance*:**

$$Z\{x(n+k)\} = z^{k}[X(Z) - \sum_{i=0}^{k-1} x(i)z^{-i}]$$
$$= z^{k}[X(Z) - x(0).z^{0} - x(1).z^{1} - ... - x(k-1).z^{-(k-1)}]$$

With : $x(0), x(1), ..., x(k-1)$ are the initial conditions.

Example 6.11:

We consider the following causal sequence:

$\{x(n)\} = \{4,8,16,...\}$

1- Write this function in the form: $x(n+k)$.

2- Find the Z transform of this function.

5) Initial and final value theorems

Initial value $s(0) = \lim_{z\to\infty} S(z)$

Final value $\lim_{n\to\infty} s(k) = \lim_{z\to 1}(z-1)S(z)$

6) Derivation theorem

$$Z\{n.x(n)\} = -z\frac{dX(z)}{dz}$$

$$Z\{n^{m}.x(n)\} = \left(-z\frac{dX(z)}{dz}\right)^{m}$$

Example 6.12:

Find the transform in Z from the sequence of a ramp:

$$r[n] = \begin{cases} n \text{ if } n \geq 0 \\ 0 \text{ if } n < 0 \end{cases}$$

The following table represents the Z transform of common discrete signals:

Laplace transform $X(p)=L\{x(t)\}$	Continuous signal $x(t)$	Sampled signal $x(n)$	Z-transformed $X(z)=Z\{x(n)\}$
1	$\delta(t)$	$\delta(\text{n})$	1
$\frac{1}{p}$	$u(t)$	$u(n)$	$\frac{z}{z-1}$
$\frac{1}{p^2}$	t	nT	$\frac{T.z}{(z-1)^2}$
$\frac{2}{p^3}$	t^2	n^2T^2	$\frac{T^2.z(z+1)}{(z-1)^3}$
$\frac{1}{p+a}$	e^{-at}	e^{-anT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(p+a)^2}$	$t.e^{-at}$	$nT.e^{-anT}$	$\frac{T.z.e^{-aT}}{(z-e^{-aT})^2}$
$\frac{b-a}{(p+a)(p+b)}$	$e^{-at}-e^{-bt}$	$e^{-anT}-e^{-bnT}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
		$a^n.u(n)$	$\frac{z}{z-a}$
		$(-a)^n.u(n)$	$\frac{z}{z+a}$
$\frac{a}{p(p+a)}$	$1-e^{-at}$	$1-e^{-anT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{w}{p^2+w^2}$	$\sin wt$	$\sin wnT$	$\frac{z\sin wT}{z^2-2z\cos wT+1}$
$\frac{p}{p^2+w^2}$	$\cos wt$	$\cos wnT$	$\frac{z(z-\cos wT)}{z^2-2z\cos wT+1}$

6.6 Inverse Z Transform

6.6.1 Inverse Z transform by polynomial division

Example 6.13:

We consider a signal $x(n)$ whose Z transform is given by the rational fraction:

$$X(z) = \frac{0.1.z^2 + 0.1.z}{z^3 - 2.6.z^2 + 2.2.z - 0.6}$$

The goal is to obtain $X(z)$ in polynomial form:

$$X(z) = X_0 + X_1.z^{-1} + X_2.z^{-2} + \ldots$$

Example 6.14:

We consider a transfer function:

$$X(z) = \frac{1}{1 - \alpha.z^{-1}} \qquad \text{For} |z| > \alpha$$

1- Find $x(n)$ by the polynomial division method.

6.6.2 Decomposition into simple elements

<u>1st case: Simple poles</u>

We have:

$$X(z) = \frac{z\left(b_0 z^{m-1} + \ldots + b_{m-1}\right)}{(z - p_1)(z - p_2)\ldots(z - p_n)}$$

So:

$$\frac{X(z)}{z} = \frac{\left(b_0 z^{m-1} + \ldots + b_{m-1}\right)}{(z - p_1)(z - p_2)\ldots(z - p_n)}$$

$$= \frac{a_1}{(z - p_1)} + \frac{a_2}{(z - p_2)} + \ldots + \frac{a_n}{(z - p_n)}$$

With:

$$a_i = (z - p_i)\frac{X(z)}{z}\bigg|_{z=+p_i}$$

So:

$$X(z) = \frac{a_1 z}{(z - p_1)} + \frac{a_2 z}{(z - p_2)} + \ldots + \frac{a_n z}{(z - p_n)}$$

Eventually:

$$Z^{-1}\{X(z)\} = x(n)$$

$$= a_1(p_1)^n + a_2(p_2)^n + \ldots + a_n(p_n)^n +$$

Example 6.15:

Let the following discrete transfer function be:

$$X(z)=\frac{10z}{(z-1)(z-0.2)}$$

1- Find the inverse Z transform of $X(z)$.

Example 6.16:

Consider the following function:

$$X(z)=\alpha k\frac{z^2}{(z-(1-k))(z-1)}$$

1- Calculate the inverse Z transform of $X(z)$.

<u>2nd case: Multiple poles</u>

$$\frac{X(z)}{z}=\frac{\left(b_0z^{m-1}+...+b_{m-1}\right)}{(z-p_1)^n(z-p_2)(z-p_3)...(z-p_n)}$$

$$=\frac{c_n}{(z-p_1)^n}+\frac{c_{n-1}}{(z-p_1)^{n-1}}+...+\frac{c_1}{(z-p_1)}+\frac{a_2}{(z-p_2)}+\frac{a_3}{(z-p_3)}+...+\frac{a_n}{(z-p_n)}$$

$$a_i=(z-p_i)\left.\frac{X(z)}{z}\right|_{z=+p_i}$$

$$\begin{cases} c_{n-j}=\dfrac{1}{j!}\left.\left(\dfrac{d^j}{dz^j}(z-p_i)^n\dfrac{X(z)}{z}\right)\right|_{z=p_1} \\ c_n=\dfrac{1}{0!}\left.\left((z-p_i)^n\dfrac{X(z)}{z}\right)\right|_{z=p_1} \\ c_{n-1}=\dfrac{1}{1!}\left.\left(\dfrac{d}{dz}(z-p_i)^n\dfrac{X(z)}{z}\right)\right|_{z=p_1} \\ c_1=\dfrac{1}{(n-1)!}\left.\left(\dfrac{d^{(n-1)}}{dz^{(n-1)}}(z-p_i)^n\dfrac{X(z)}{z}\right)\right|_{z=p_1} \end{cases}$$

Example 6.17:

Consider the following function:

$$X(z)=\frac{1}{z^2(z-0.5)}$$

1- Calculate the inverse Z transform of $X(z)$.

Evaluation of $x(n)$ For $n=0,1,2$

$$\begin{cases} x(0)=8-8=0 \\ x(1)=8(0.5)-4=0 \\ x(2)=8(0.5)^2-2=0 \end{cases}$$

So we can write:

$$x(n = \begin{cases} (0.5)^{n-3}; & n \geq 3 \\ 0; & n < 0 \end{cases}$$

We can use the delay theorem $Z\{g(n-k)\} = z^{-k}G(z)$, writing:

$$X(z) = \frac{z}{(z-0.5)} z^{-3}$$

6.6.3 The convolution product and the Z transform

$$Z\{s(n) = e(n) * h(n)\} = E(z).H(z)$$

Example 6.18:

Let the following two functions be:

$$e(n) = 2\delta(n) - 3\delta(n-2) + 4\delta(n-3)$$
$$h(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$$

1- Calculate $s(n) = e(n) * h(n)$.

6.7 Solving difference equations with the ZT

Example 6.19:

Let the equation with the following difference:

$$x(n+2) - 3.x(n+1) + 2x(n) = \delta(n)$$

The initial conditions:

$$\begin{cases} x(0) = 0 \\ x(1) = 0 \end{cases}$$

1- Solve this equation using the Z transform.