Chapter 01

General information on signals

1.1 Introduction

Signal theory is a set of mathematical tools that allows us to describe, classify and better exploit signals. Signal processing is an essential discipline these days. Its purpose is the development or interpretation of signals carrying information. Its goal is therefore to succeed in extracting a maximum of useful information from a signal disturbed by noise by relying on the resources of electronics and computer science.

1.1 Definition of a signal

A signal is the physical representation of information, which it conveys from its source to its recipient. It provides the means to analyze, design and characterize information processing systems.

- The physical quantities are (electrical v, i, E …, mechanical F, Ce …, chemicals T …etc).
- The mathematical description of signals is the goal of signal theory. $\left\{\begin{array}{l}\text{Amplitude}\\ \text{Frequency}\end{array}\right.$

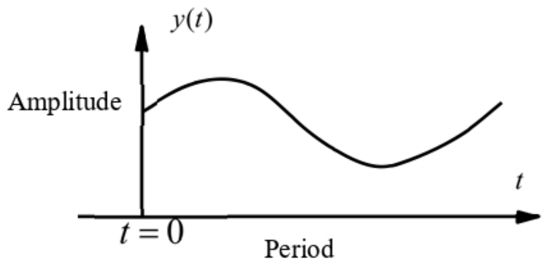


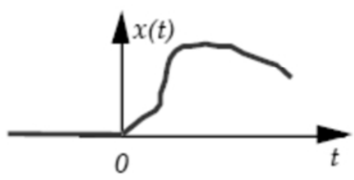
Fig. 1.1: Analog signal.

1.2 Signal Definitions

1.2.1 Causal signal

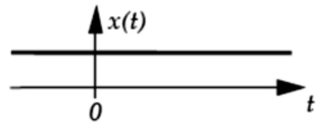
A signal is causal if it is zero for all negative instants

$x(t)=0\ \forall\ t<0$



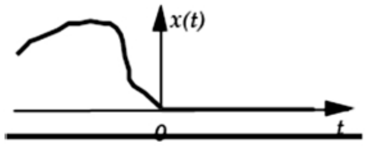
1.2.2 Non-causal signal

A signal is non-causal if it is not zero for some or all negative instants.



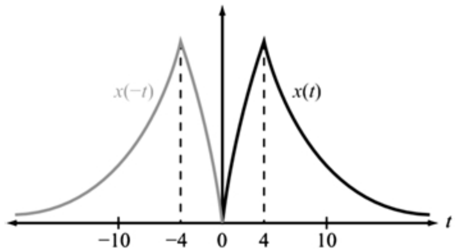
1.2.3 Anti-causal signal

A signal is anti-causal if it is zero for all positive times.



1.2.4 Time inversion

Graphically, $s(t) = x(-t)$ corresponds to the symmetrized version of $x(t)$ relative to the y-axis.



1.2.5 Time Shift

- If $t_0 > 0$, we operate an advance: horizontal translation of t_0 to the left.
- If $t_0 < 0$, we operate a delay: horizontal translation of t_0 to the right.

Graphically, the advanced signal $x(t+t_0)$ or delayed $x(t-t_0)$ corresponds to the horizontally translated version of the original signal $x(t)$.

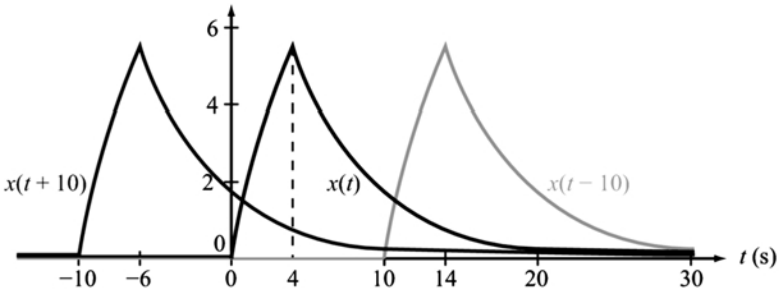


Fig. 1.2: time lag.

1.2.6 Change of scale

Graphically, $y(t) = x(at)$ (with a is the scale factor).

If $0 < a < 1$, $y(t)$ is an expanded version of $x(t)$; If $a > 1$, $y(t)$ is a compressed version of $x(t)$.

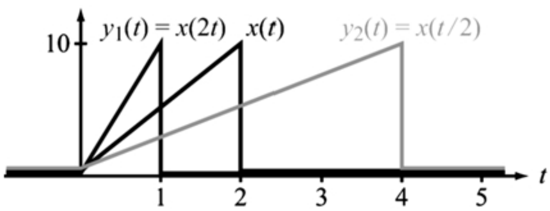


Fig. 1.3: change of scale.

1.2.7 Even signal

$x(-t) = x(t)\ \forall\, t$, Graphically, an even signal has symmetry about the y-axis.

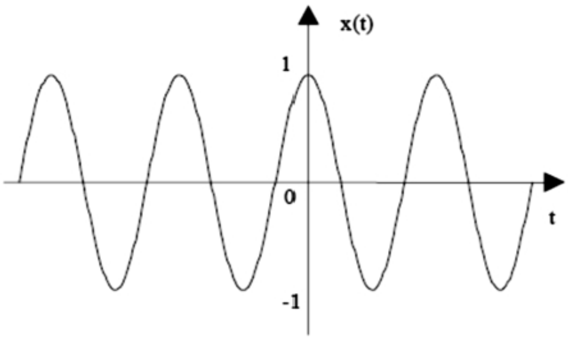


Fig. 1.4: even function.

1.2.8 Odd signal

$x(-t) = -x(t)\ \forall\, t$, Graphically, an odd signal has symmetry about the origin

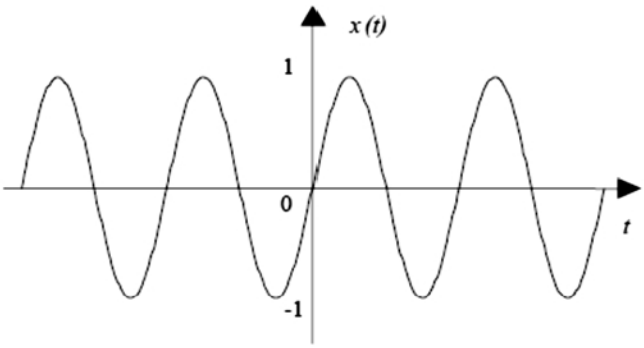
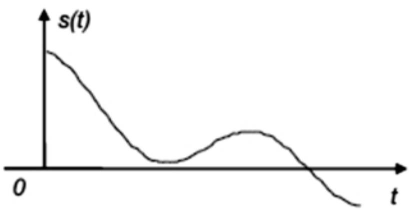
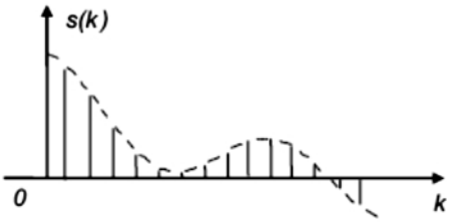


Fig. 1.5: odd function.

1.3 Signal classifications

1.3.1 Category 01

The signal is considered as a function of time according to the variable 't', we have 2 categories of signals:

Analog signals (continuous)	Digital signals (discrete)
They are usually described mathematically in the form of a function $s(t)$ where the variable t is associated with time.	Unlike analog signals, digital signals are not continuous, but discrete in value and time.
 $[\forall\, t, \exists\ y(t)]$	 $[\exists\ y(t)$ for some 't' $]$

1.3.2 Category 02

We consider the nature of the evolution of the signal as a function of time, so we can notice that there are 2 categories of signals:

$$\begin{cases} \text{Determinist} \\ \text{Random} \end{cases}$$

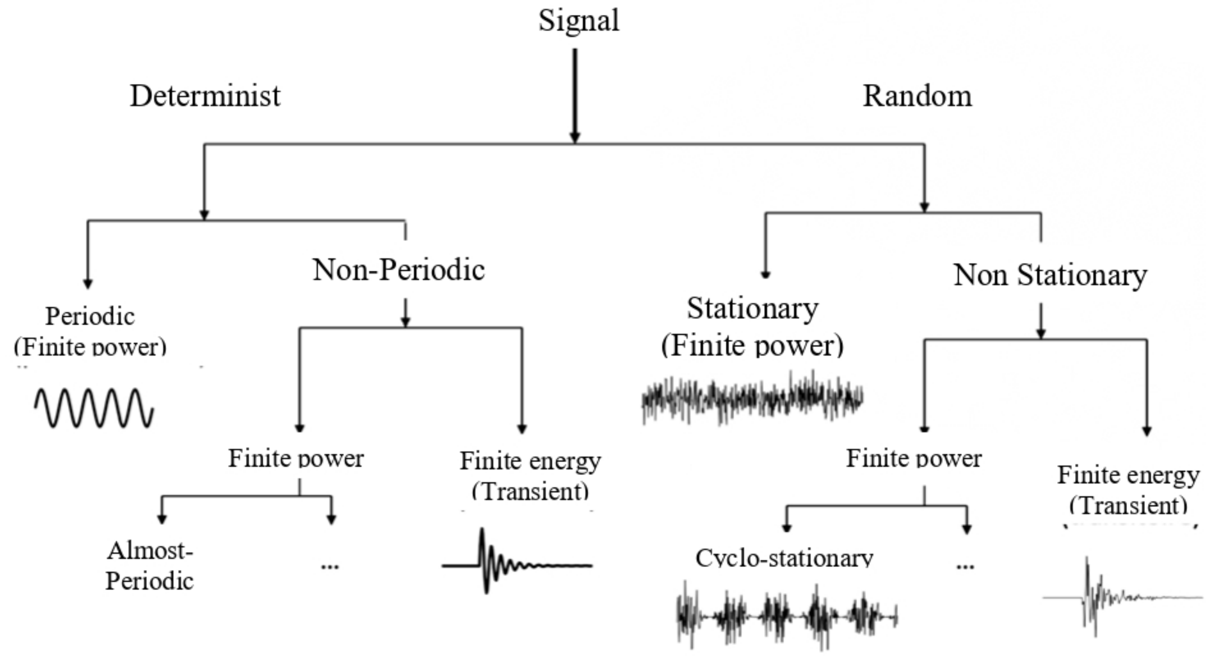


Fig. 1.6: Classification of signals.

A. Deterministic signals:

The evolution over time can be perfectly described by a mathematical model [$\sin(t)$] :

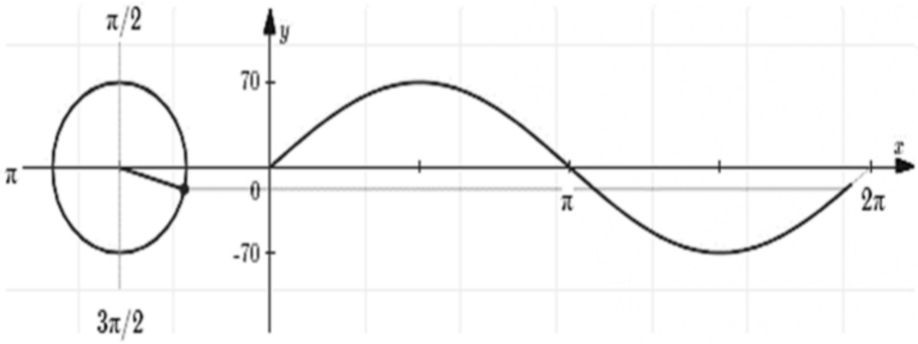


Fig. 1.7: Sinusoidal signal (deterministic).

These signals come from phenomena for which we know the corresponding physical laws and the initial conditions, thus allowing us to predict the result. To describe the evolution of deterministic signals, we use deterministic models which are based on the usual mathematical functions which allow us to perfectly predict their evolution (Fig.1.7).

B. Random signals:

Temporal behavior is unpredictable and for the description of which one must be satisfied with statistical observations. To describe the evolution of random signals, one uses probabilistic models that are based on probability theory and statistics (Fig.1.8).

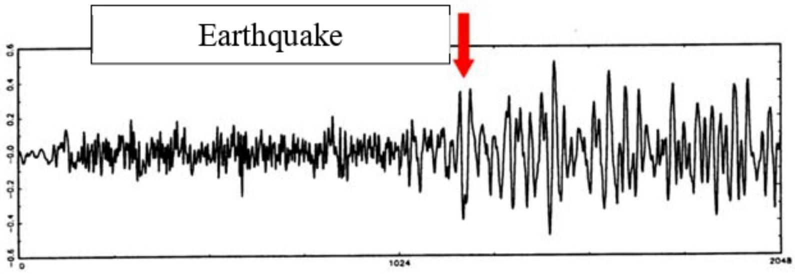


Fig. 1.8: Random signals.

1.3.3 Classification and energy study of signals

A. Classification methods

A.1 Phenomenological classification
- Determinist
- Random

A.2 Morphological classification
- Continuous (analog)
- Discrete (digital)

A.3 Spectral classification
- High frequencies
- Base frequencies

A.4 Dimensional classification ⟶ Number of variables (2,3, …)

A.5 Energy Classification
- Finite energy signals (transient signals)
- Finite average power signals (periodic signals)

B. Characteristic values of analog signals

B.1 Average value

- Non-periodic signal:

$$\bar{S} = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} s(t)dt$$

- Periodic signal of period T_0 :

$$\bar{S} = \frac{1}{T_0} \int_{0}^{T_0} s(t)dt$$

B.2 Effective value

- Non-periodic signal:

$$S_{eff}^2 = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} s^2(t)dt$$

- Periodic signal of period T_0 :

$$S_{eff}^2 = \frac{1}{T_0} \int_{0}^{T_0} s^2(t)dt$$

Example 1.1: Let a sinusoidal signal of period T_0 such as : $s(t) = \sin\left(\frac{2\pi}{T_0}t\right)$

1- Graph this signal.

2- Calculate the mean value and the effective value of this signal:

C. Energy classification

In electricity: the instantaneous power is: $p(t) = v(t)i(t)\ [W = V.A]$

$$v(t) = R.i(t) \Rightarrow p(t) = R.i(t)^2 = \frac{1}{R}.v(t)^2$$

For $R = 1\Omega \Rightarrow p(t) = i(t)^2 = s(t)^2$ (Signal square).

C.1 Total energy

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2\, dt$$

C.2 Total average power

- Non-periodic signal: $P_s = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |s(t)|^2\, dt$

- Periodic signal of period T_0 : $P_s = \frac{1}{T_0} \int_{0}^{T_0} |s(t)|^2\, dt$

- **Finite energy signals** or at zero average power: $E_s = \int_{-\infty}^{+\infty} |s(t)|^2\, dt < \infty$

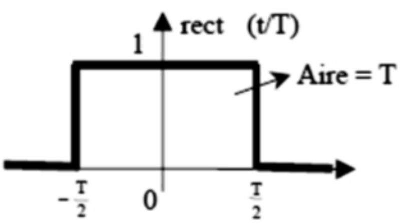
- **Finite average power signals** non-zero (infinite energy): $0 < P_s = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |s(t)|^2\, dt < \infty$

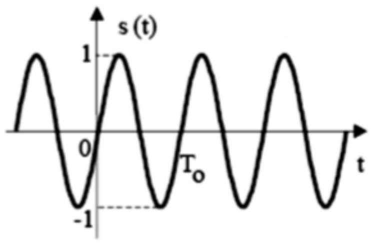
Example 1.2:

1- Calculate the energy and power of the signal from example 1.1.

2- Graphically represent the signal $|s(t)|^2$.

Exercise 01 (homework): either the following 2 signals:





1. What is the deference between the T of the gate signal and the T_0 of the sinusoidal signal.
2. Calculate the energy and power of the gate signal.

Exercise 02 (homework): either the following 2 signals:

$x(t) = \sin(2.\pi.f_0 t)$ And $y(t) = \cos(2.\pi.f_0 t + \varphi)$

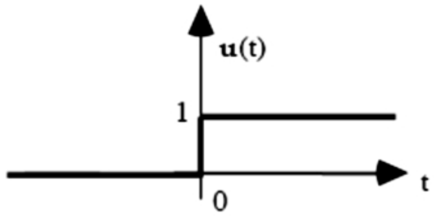
1. Calculate the energy of these signals.
2. Calculate the interaction power $P_{xy} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t).y(t).dt$.
3. Draw the curve P_{xy} depending on φ .

1.4 Common signals

1.4.1 The step function

$$u(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ t \geq 0 \end{cases}$$

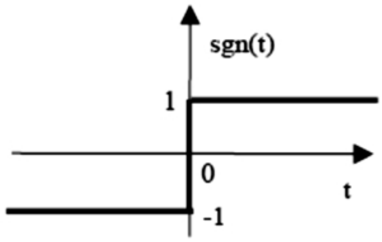
$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$



1.4.2 the sign function

$$\operatorname{sgn}(t) = \begin{cases} 1 & \forall\ t > 0 \\ -1 & \forall\ t < 0 \\ 0 & t = 0 \end{cases}$$

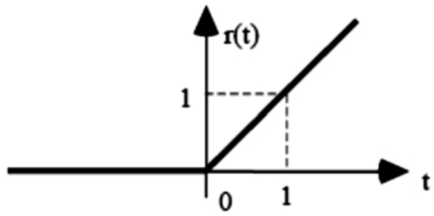
$$\operatorname{sgn}(t) = 2.u(t) - 1 = \frac{t}{|t|} (with\, t \neq 0)$$



1.4.3 The ramp function

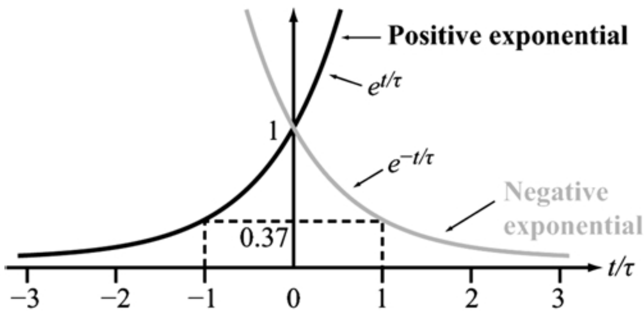
$$r(t) = \begin{cases} 0 & if\ t < 0 \\ t & if\ t \geq 0 \end{cases}$$

$$r(t) = t.u(t) = \int u(t) \Rightarrow u(t) = \frac{dr(t)}{dt} (with\, t \neq 0)$$

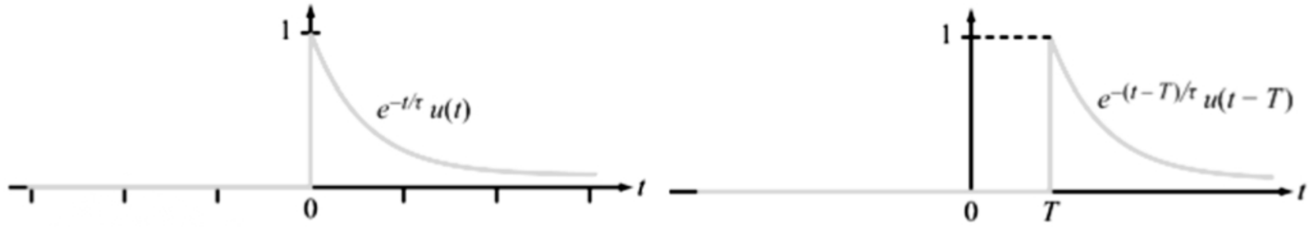


1.4.4 Real-valued exponential signal

$$s(t) = A.e^{\frac{t}{\tau}} \quad or \quad s(t) = A.e^{-\frac{t}{\tau}}$$



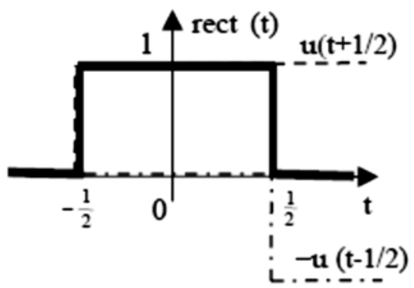
(c) Multiplication of $e^{-t/\tau}$ by $u(t)$



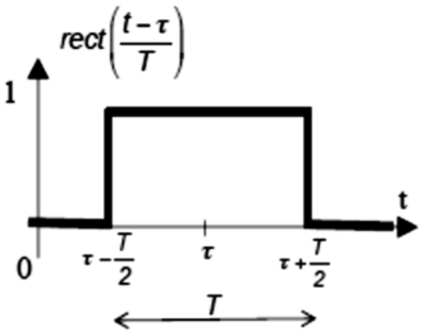
1.4.6 The rectangular function

$$p_1(t) = rect(t) = \begin{cases} 1 & \forall \quad |t| \le \frac{1}{2} \\ 0 & \forall \quad |t| > \frac{1}{2} \end{cases}$$

$$p_1(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$$



$$p_T(t-\tau) = rect(\frac{t-\tau}{T}) = \begin{cases} 0 & \forall \quad t < \tau - \frac{T}{2} \\ 1 & \forall \quad \tau - \frac{T}{2} \le t \le \tau + \frac{T}{2} \\ 0 & \forall \quad t > \tau + \frac{T}{2} \end{cases}$$

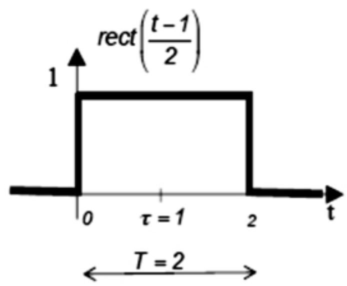


Or : T : duration of the rectangular window;

τ : location of the center of the window.

Example 1.3:

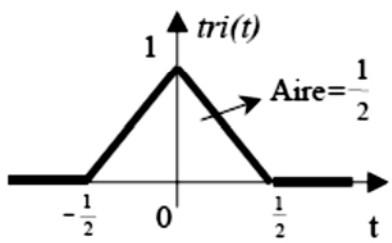
1. Find the relationship between the signal $p_T(t-\tau)$ and the signal $u(t)$.
2. Write the mathematical expression of this signal and their relationship to the step function.



1.4.7 The triangular function

$$tri_1(t) = \begin{cases} 1-2|t| & \forall \quad |t| \le \frac{1}{2} \\ 0 & \forall \quad |t| > \frac{1}{2} \end{cases}$$

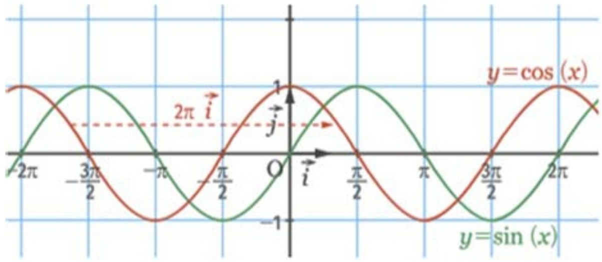
$$tri_1(t) = 2.r\left(t+\frac{1}{2}\right) - 4.r(t) + 2.r\left(t-\frac{1}{2}\right)$$



1.4.8 Sinusoidal functions

$x(t) = A.\sin(w_0.t)$

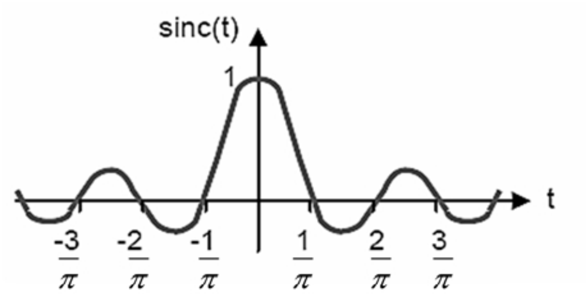
$y(t) = A.\cos(w_0.t) = A.\sin(w.t + \varphi)$



A : the amplitude; T_0 : the period [s] ; $f_0 = \frac{1}{T_0}$: the frequency [$Hz(s^{-1})$] ; φ : the phase shift [rad] ; w_0 : pulsation (angular velocity) [$\frac{rad}{s}$] ; $w_0.t$ the angle [rad].

1.4.9 The cardinal sin function

The cardinal sin function is obtained by performing the ratio of a sinusoidal function and its argument is defined by:

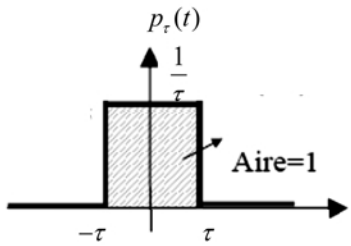


$$\sin c(t) = \frac{\sin(t)}{t}$$

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \sin c(t) = 1 \; and \; \sin c(0) = 1$$

1.4.10 The Dirac impulse

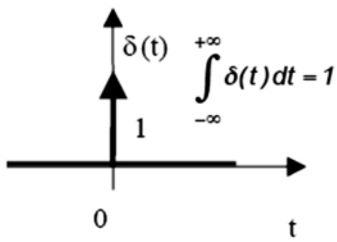
Consider the limit of the rectangular window of width τ below when τ tends towards 0.

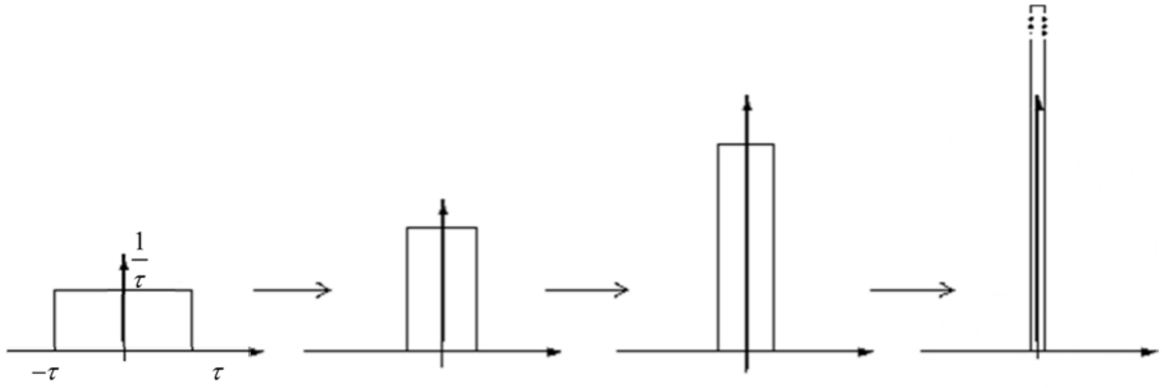


$$\lim_{\tau \to 0} \frac{1}{\tau} p_\tau(t) = \delta(t)$$

- Cannot graph this function.

- The graphical representation of a Dirac pulse is a vertical arrow (with area=1) placed in $t = 0$ of height proportional to the weighting constant here equal to 1 (the surface of the rectangular).





A. Properties of the Dirac impulse

$$\int_{-\infty}^{+\infty} s(t)\delta(t-t_0)dt = s(t_0); \quad \int_{-\infty}^{+\infty} s(t)\delta(t)dt = s(0) \quad if\ t_0 = 0$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t); \qquad \int_{-\infty}^{t} \delta(t)dt = 1 \qquad if\ s(t) = 1\ and\ t_0 = 0$$

$$s(t)\delta(t-t_0) = s(t_0)\delta(t-t_0); \quad s(t)\delta(t) = s(0)\delta(t) \quad if\ t_0 = 0$$

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

1.4.11 Dirac's comb

A Dirac comb is a sequence of Dirac pulses repeating on the time axis with a period T_0 :

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{+\infty} \delta(t = kT_0); \quad k \in \mathbb{Z}$$